

D-brane Charges in Gravitational Duals of $2 + 1$ Dimensional Gauge Theories and Duality Cascades

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Abstract

We perform a systematic analysis of the D-brane charges associated with string theory realizations of $d = 3$ gauge theories, focusing on the examples of the $\mathcal{N} = 4$ supersymmetric $U(N) \times U(N + M)$ Yang-Mills theory and the $\mathcal{N} = 3$ supersymmetric $U(N) \times U(N + M)$ Yang-Mills-Chern-Simons theory. We use both the brane construction of these theories and their dual string theory backgrounds in the supergravity approximation. In the $\mathcal{N} = 4$ case we generalize the previously known gravitational duals to arbitrary values of the gauge couplings, and present a precise mapping between the gravity and field theory parameters. In the $\mathcal{N} = 3$ case, which (for some values of N and M) flows to an $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory in the IR, we argue that the careful analysis of the charges leads to a shift in the value of the B_2 field in the IR solution by $1/2$, in units where its periodicity is one, compared to previous claims. We also suggest that the $\mathcal{N} = 3$ theories may exhibit, for some values of N and M , duality cascades similar to those of the Klebanov-Strassler theory.

1 Introduction

There has been significant recent progress in the study of superconformal field theories in 2+1 dimensions, following the discovery of a Lagrangian for a theory with $\mathcal{N} = 8$ supersymmetry by Bagger, Lambert, and Gustavsson [1,2]. The discovery of this theory is significant in light of the fact that a Lagrangian formulation for theories with these properties was believed not to exist [3]. The original formulation of these new models used an exotic mathematical structure called a “3-algebra.” Subsequently, it was shown that these theories admit a formulation in terms of an ordinary Chern-Simons gauge theory with an $SU(2) \times SU(2)$ gauge group and bi-fundamental matter [4]. These theories are closely related to the decoupled theories living on the worldvolume of M2-branes in M theory.

An interesting class of generalizations with $\mathcal{N} = 6$ supersymmetry, that describes the decoupled theory on N_2 M2-branes at a $\mathbf{C}^4/\mathbf{Z}_k$ orbifold point, and that has a known gravitational dual, was identified by Aharony, Bergman, Jafferis, and Maldacena (ABJM) [5]. It arises from a brane construction consisting of N_2 D3-branes on $\mathbf{R}^{1,2} \times S^1$ intersecting with NS5 and $(1, k)$ 5-brane defects. After taking a decoupling limit, the result is a Chern-Simons theory with gauge group $U(N) \times U(N)$ with $N = N_2$ and with Chern-Simons terms of levels k and $-k$ for the two gauge groups, coupled to scalar and fermion bifundamental fields. The supersymmetry is enhanced to $\mathcal{N} = 8$ for $k = 1, 2$. The theory has a dual gravitational formulation as M theory on a \mathbf{Z}_k orbifold of $AdS_4 \times S^7$. For sufficiently large values of k , it is natural to describe this model in a type IIA description by reducing along the Hopf fiber of the S^7 . The background geometry is then $AdS_4 \times \mathbf{CP}^3$, with some RR fluxes.

A generalization of ABJM adding fractional branes, with gauge group $U(N) \times U(N+M)$ (where $N = N_2$, $M = N_4$ and $N_4 \leq k$), was considered by Aharony, Bergman, and Jafferis (ABJ) [6]. According to ABJ, the dual M theory description of this model is the same $AdS_4 \times S^7/\mathbf{Z}_k$ geometry, with N_4/k units of discrete (torsion) 3-form flux through an S^3/\mathbf{Z}_k cycle in S^7/\mathbf{Z}_k . The type IIA description of this background was claimed to be $AdS_4 \times \mathbf{CP}^3$, with an NS-NS 2-form field through the \mathbf{CP}^1 2-cycle in \mathbf{CP}^3 , equal to $b = -N_4/k$ (in units where its periodicity is one¹).

Several consistency tests for this conjecture were presented in [6]. There are, however, some subtle and confusing aspects to the proposal of ABJ. In most constructions of gravity duals involving product gauge groups, the flux of the B_2 field through the 2-cycle is continuous. In the brane construction of these theories [7,8] it parameterizes the distance between NS5-branes, which is directly related to the relative strength of the gauge couplings. The fact that the gauge couplings have decoupled by the time one flows to the superconformal

¹The minus sign is the result of the conventions used in this paper, summarized in appendix B.

theory in the IR assures us that there is no immediate contradiction with the discreteness of b in the supergravity dual of the IR fixed point. Nonetheless, it would be interesting to understand how this decoupling manifests itself in the dual gravitational description of the renormalization group (RG) flow, when the superconformal field theory in the IR is embedded into a supersymmetric Yang-Mills-Chern-Simons-Matter theory. It appears that some form of attractor mechanism is at work, with b flowing from a continuous value in the UV to a discrete value in the IR.

Another slightly counterintuitive aspect of the proposal of ABJ is that the flux of the *potential field* B_2 , rather than that of its *field strength*, is quantized and related to integer charges characterizing the system. This is not an entirely unfamiliar notion. Similar quantizations of potentials also appear in the near-horizon limit of a D1-D3 bound state for a D3-brane wrapping a T^2 [9]. The general mechanism for quantization of a gauge potential arises when there are Chern-Simons terms in the action, mixing gauge potentials and gauge field strengths. Such Chern-Simons terms are certainly part of supergravities in 10 and 11 dimensions.

Chern-Simons terms introduce various subtleties in measuring and quantizing charges. In simple situations, we usually expect charges to satisfy all of the following four properties:

- Localization: we expect elementary sources of charge to be point-like (or localized on branes) and to respect Gauss' law.
- Gauge invariance: we expect charge to be a physical, gauge-invariant notion.
- Quantization: we expect charges to respect a Dirac quantization condition.
- Conservation: we expect charges to be an invariant quantity in dynamical processes.

As reviewed by Marolf [10], in the presence of Chern-Simons terms there is in general no definition of charge which simultaneously respects all four of the properties listed above. Instead, there are three different notions of charge, which Marolf called Maxwell, Page, and Brane charges, each of which satisfies a subset of the properties above, and which can take different values. So, care is needed when discussing issues such as quantization of charges and enumeration of gauge equivalence classes of physical configurations.

In this article, we will closely examine the D-brane charges and the related quantization of the potential field in the models of ABJM and ABJ [5, 6], and also in the $\mathcal{N} = 3$ gauge theories obtained by adding Yang-Mills kinetic terms to the ABJM and ABJ theories. Most of the relevant issues arise already when looking at the same theories with $k = 0$ (no Chern-Simons coupling), which leads to $\mathcal{N} = 4$ supersymmetric gauge theories. Section 2 is devoted

to analyzing this case (including also the possibility of having additional hypermultiplets in the fundamental representation of one of the gauge groups). In section 3, we investigate the quantization conditions of charges and potentials in the ABJM and ABJ models. Broadly speaking, we confirm the basic picture of ABJ and ABJM where the gauge potentials take quantized values in the IR, depending on the quantized charges. However, we find a subtle half-integer shift in b , related to the Freed-Witten anomaly [11], which corrects the specific quantized value of the gauge potential field.² We verify that the consistency tests carried out in ABJ and ABJM are compatible with this correction.

Most of the subtleties in charge quantization are already present in the D1-D3 bound state system in type IIB string theory with some NS-NS B_2 -field in the background. We review this setup in appendix A.

The same subtleties in defining charges appear also in the Klebanov-Strassler theory [13], where they are an essential part of a “duality cascade.” In the corresponding gravitational backgrounds, the D3-brane Maxwell charge changes during the RG flow (corresponding to a reduction in the rank of the gauge group when flowing to the IR), while the Page charge is quantized. The Page charge is not invariant under large gauge transformations, and these transformations may be viewed as Seiberg duality transformations that relate the IR behavior of different field theories. The cascade of [13] may be viewed as a sequence of such duality transformations. The $\mathcal{N} = 3$ theories discussed above also have a Seiberg-like duality in the IR [6], which in the brane construction looks very similar to the one of [13]; thus, it is natural to suggest that these theories may also exhibit duality cascades. We discuss this possibility, and its implications for the IR behavior of theories with general values of N_2 and N_4 , in section 4. Appendix B contains a summary of our conventions.

2 A $d = 3$ $\mathcal{N} = 4$ theory with fractional branes and flavors

2.1 The dual gravitational background

An interesting $d = 3$ gauge theory with a known gravitational dual is the $\mathcal{N} = 4$ supersymmetric gauge theory arising from the low-energy theory on N_2 D2-branes and N_6 D6-branes in type IIA string theory, oriented as

$$\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 \text{D2} & \bullet & \bullet & \bullet & & & & & & & \\
 \text{D6} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & &
 \end{array} \tag{2.1}$$

²This shift also contributes to the correction to the radius of AdS_4 due to the effects of curvature corrections and discrete torsion discussed in [12].

where the coordinates 3456 span an ALE space on which the D6-branes are wrapped. This configuration preserves eight supercharges, and when lifted to M theory it describes N_2 M2-branes localized in an 8-dimensional transverse geometry of the form $ALE \times TN_k$, with $k = N_6$ [14]. The action for supergravity in 11 dimensions takes the standard form³

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{2}|G_4|^2 \right) - \frac{1}{2\kappa_{11}^2} \int \frac{1}{6} C_3 \wedge G_4 \wedge G_4, \quad (2.2)$$

which can be solved by an ansatz of the form

$$ds^2 = H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}(ds_{ALE}^2 + ds_{TN_k}^2), \quad (2.3)$$

$$G_4 = dC_3 = dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1}, \quad (2.4)$$

where the Taub-NUT metric is given by

$$ds_{TN_k}^2 = V(r)^{-1}(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) + V(r)R_{11}^2 k^2 \left(d\psi - \frac{1}{2} \cos \theta d\phi \right)^2 \quad (2.5)$$

with

$$V(r) \equiv \left(1 + \frac{kR_{11}}{2r} \right)^{-1}, \quad R_{11} = g_s l_s, \quad (2.6)$$

for the range of coordinates $0 \leq r < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 2\pi/k$. The field equations imply that the warp factor H is a solution of the Poisson equation

$$\nabla_{TN}^2 H(\vec{y}, \vec{r}) + \nabla_y^2 H(\vec{y}, \vec{r}) = -(2\pi l_p)^6 N_2 \delta^4(\vec{y}) \delta^4(\vec{r}), \quad (2.7)$$

where \vec{y} denotes the ALE coordinates and \vec{r} denotes the Taub-NUT coordinates, with a source coming from N_2 M2-branes at the origin⁴. This equation is separable in y and r , where y is the radial variable of the ALE space, and the solution can be written in the form of a convolution of a Bessel function and a confluent hypergeometric function [14]. If we take the ALE space to be the $\mathbf{R}^4/\mathbf{Z}_2$ orbifold, as we will assume from here on, this geometry describes the supergravity dual of a $U(N_2) \times U(N_2)$ gauge theory, with two bifundamental hypermultiplets and with $k = N_6$ hypermultiplets in the fundamental representation of one of the $U(N_2)$ gauge groups. (The issue of which one of the two will be discussed below.) This is easiest to see in the T-dual brane construction, that we will describe in section 2.5 below. Of course, the M theory description is only useful at low energies, but the same solutions (reduced on the M theory circle, as we will describe below) are valid also in type IIA supergravity for higher energies. This class of backgrounds is particularly useful to

³See Appendix B for a summary of our conventions.

⁴In this section we assume that the charges are large, and we neglect the corrections to the charges coming from the curvature.

visualize the Intriligator-Seiberg mirror symmetry [15] for field theories in $2 + 1$ dimensions from the point of view of the holographic dual⁵ [14].

Generalizing this construction to the case where we include also N_4 fractional branes, which are (in the type IIA description) D4-branes wrapped on the vanishing 2-cycle of the ALE space, is relatively straightforward. If we are interested in the case where all of the D2, D4, and D6-branes are coincident in the type IIA description, we can take the ansatz

$$ds^2 = H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}(ds_{ALE}^2 + ds_{TN_k}^2), \quad (2.8)$$

$$G_4 = dC_3 = dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1} + G_4^{SD}, \quad (2.9)$$

$$G_4^{SD} = d(lV\omega_2 \wedge \sigma_3 + 2\alpha\omega_2 \wedge d\psi) \quad (2.10)$$

for some constants l and α , where

$$\frac{1}{2}\sigma_3 \equiv d\psi - \frac{1}{2}\cos\theta d\phi, \quad (2.11)$$

and ω_2 is the self-dual 2-form dual to the collapsed 2-cycle in the ALE space, normalized so that⁶

$$\int_{ALE} \omega_2 \wedge \omega_2 = \frac{1}{2}. \quad (2.12)$$

We denote the 2-cycle dual to ω_2 by Σ , and the non-compact 2-cycle which is dual to Σ on the ALE space by \mathcal{M} . Then, it follows that

$$\int_{\Sigma} \omega_2 = 1, \quad \int_{\mathcal{M}} \omega_2 = \frac{1}{2}. \quad (2.13)$$

The four-form G_4^{SD} is a self-dual 4-form on $ALE \times TN_k$. We will fix the parameters l and α in terms of the field theory data shortly.

To express this ansatz in terms of type IIA supergravity, let $dx_{11} = kR_{11}d\psi$ and perform the standard reduction. The ansatz can then be expressed in the form

$$ds_{IIA}^2 = H^{-1/2}V^{1/2}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/2}V^{1/2}ds_{ALE}^2$$

⁵One can also consider taking a $TN_{k_1} \times TN_{k_2}$ space-time. Reducing to type IIA and T-dualizing implies that this corresponds to a field theory on N_2 D3-branes intersecting k_1 NS5-branes and k_2 D5-branes, although the supergravity description only captures the aspect of the theory where the positions of the defects are smeared. There is a decoupling limit in which this describes a $3 + 1$ dimensional gauge theory with defects at the positions of the brane intersections, and a further decoupling limit where we reduce to the low-energy $2 + 1$ dimensional gauge theory.

⁶This convention is used in [16].

$$+H^{1/2}V^{-1/2}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \quad (2.14)$$

$$A_1 = -\frac{1}{2}R_{11}k \cos\theta d\phi, \quad (2.15)$$

$$A_3 = -(H^{-1} - 1)dt \wedge dx_1 \wedge dx_2 - lV\omega_2 \wedge \cos\theta d\phi, \quad (2.16)$$

$$B_2 = -\frac{2}{R_{11}k}(lV\omega_2 + \alpha\omega_2), \quad (2.17)$$

$$e^\phi = g_s H^{1/4} V^{3/4}. \quad (2.18)$$

It is convenient to introduce a field variable b by the relation

$$B_2 = (2\pi)^2 \alpha' b \omega_2, \quad (2.19)$$

such that a large gauge transformation of B_2 takes $b \rightarrow b + 1$.

The equations of motion of 11 dimensional supergravity are satisfied provided H obeys

$$0 = (\nabla_y^2 + \nabla_{TN}^2) H + \frac{l^2 V^4}{2r^4} \delta^4(\vec{y}) + (2\pi l_p)^6 Q_2 \delta^4(\vec{y}) \delta^4(\vec{r}), \quad (2.20)$$

which can be inferred most efficiently from

$$d *_{11} G_4 = \frac{1}{2} G_4 \wedge G_4 + (2\pi l_p)^6 Q_2 \delta^4(\vec{y}) \delta^4(\vec{r}) d^4 \vec{y} \wedge d^4 \vec{r}. \quad (2.21)$$

Here we allowed some arbitrary source of M2-brane charge sitting at the origin, with a magnitude Q_2 that will be determined below.

Upon reduction to type IIA, equation (2.20) reads

$$0 = \left(1 + \frac{kR_{11}}{2r}\right)^{-1} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}\right) H(y, r) + \nabla_y^2 H(y, r) + \frac{l^2 V^4}{2r^4} \delta^4(\vec{y}) + (2\pi l_s)^5 g_s Q_2 \delta^4(\vec{y}) \delta^3(\vec{r}). \quad (2.22)$$

In type IIA string theory we interpret the source as coming from D2-branes, D4-branes and D6-branes sitting at the origin, so we have

$$Q_2 = N_2 + b_0 N_4 + \frac{N_6 b_0^2}{4}, \quad (2.23)$$

where b_0 is the value of the b field at the position of the branes, and the source arises from the presence of the Chern-Simons coupling $e^{F+B} \wedge C$ on the worldvolume of the D-branes, where C is the sum of the RR potentials A_i (we assume that no field strengths are turned on in the D-brane worldvolumes). Assuming that $H(r, y)$ is solved with respect to appropriate sources, and that the parameters α and l are set to appropriate values, this background, in the suitable decoupling limit, is dual to the $U(N_2 + N_4) \times U(N_2)$ quiver gauge theory with N_6 fundamental matter multiplets.⁷ These solutions are 2 + 1 dimensional versions of

⁷Since 2+1 dimensional gauge theories are always asymptotically free, the dual gravitational backgrounds are always highly curved in the UV region (large radial coordinate). Our discussion of these backgrounds will be limited to smaller values of the radial coordinate, where the solutions are weakly curved.

a construction presented in [16, 17]. In fact, precisely this background was constructed also in [18], with one important caveat: [18] considered only the case where

$$b_\infty \equiv b(r \rightarrow \infty) = \frac{1}{2}. \quad (2.24)$$

As we will see shortly, allowing b_∞ to take other values provides opportunities to explore many interesting physical features of this supergravity background.

2.2 Charge quantization

To construct the gravitational background dual to the $U(N_2 + N_4) \times U(N_2)$ gauge theory we still need to fix the values of the parameters α and l . These parameters, which characterize the four form G_4^{SD} , are naturally related to the number of fractional branes N_4 , and to the value of the b field at infinity, which is related [7, 8] to the ratio between the gauge couplings of the two groups. In fact, we see immediately from the fact that $V(r) \rightarrow 1$ as r is taken to be large, that

$$l = -2\pi^2 k l_s^2 R b_\infty - \alpha. \quad (2.25)$$

All that remains is to extract one more linear relation relating l , α , and N_4 .

However, there is one small complication in specifying these parameters. Recall that the gauge-invariant 4-form field strength in type IIA supergravity is given by

$$\tilde{F}_4 = dA_3 + dB_2 \wedge A_1. \quad (2.26)$$

Since the D4-branes are wrapped on Σ , their flux will thread $\mathcal{M} \times S^2$ where S^2 is the sphere surrounding the D6-branes. Using our solution above, the gauge-invariant flux through $\mathcal{M} \times S^2$ at some fixed r is then

$$\int_{\mathcal{M} \times S^2} (-\tilde{F}_4) = -2\pi l V(r), \quad (2.27)$$

which depends continuously on r .

On one hand, this is directly analogous to the continuous evolution of the charge in the Klebanov-Strassler construction [13], which was interpreted as a cascade of dualities. It is natural to expect that the presence of fractional branes in the $2 + 1$ dimensional context may give rise to some similar duality cascade. On the other hand, it is peculiar for charges, which are generally quantized and conserved, to vary continuously as the Gaussian surface surrounding the sources is varied.

An elegant resolution to this apparent conflict is reviewed by Marolf in [10]. There are three similar, yet distinct, notions of charges: Maxwell charge, Page charge, and brane

charge. We will illustrate the difference between these charges in the example of the D4-brane charge in type IIA string theory which was mentioned above.

Maxwell charge is the flux of a gauge-invariant field strength (such as (2.26)), measured at infinity; in the example above it is the $r \rightarrow \infty$ limit of (2.27). The corresponding current is defined by

$$-d\tilde{F}_4 = *j_5^{Maxwell}. \quad (2.28)$$

From the definition it is clear that this charge is gauge-invariant and conserved. However, in general it is not quantized, and in the presence of Chern-Simons-type terms in the bulk, it obtains contributions from these terms as well as from the localized sources; recall that in the absence of sources the Bianchi identity for \tilde{F}_4 takes the form $d\tilde{F}_4 = -F_2 \wedge H_3$. This is what allows this charge to continuously depend on the radial coordinate, as in (2.27).

Brane charge, on the other hand, is the charge coming purely from the localized sources, such as (2.23) in the case of D2-brane charge. In the D4-brane example it is defined by

$$-d\tilde{F}_4 - F_2 \wedge H_3 = *j_5^{brane}. \quad (2.29)$$

The definition implies that it is gauge-invariant, but in general it is not conserved or quantized. The localized sources contributing to the brane charge can come from various branes that carry the appropriate charge. In our example it takes the form $*j_5^{brane} = *j_5^{D4} + B_2 \wedge *j_7^{D6}$, where the first term is the D4-brane source current, including contributions from gauge fields on D6-branes $F_2^{D6} \wedge *j_7^{D6}$ which count D4-branes in bound states with D6-branes, while the second term is a D4-brane current induced on D6-branes in the presence of a B_2 field. These brane currents are normalized such that for a D-brane spanning the directions $x_0 \dots x_p$,

$$j_{p+1}^{Dp} = (2\pi l_s)^{7-p} g_s \delta^{9-p}(\vec{y}) dx_0 \wedge dx_1 \dots \wedge dx_p. \quad (2.30)$$

We expect that a quantized charge should just measure the (integer) number of D4-branes, and the correct charge that does this in the presence of Chern-Simons terms (and respects Gauss' law, in that it has the same value for any surface surrounding the sources) is the Page charge. The Bianchi identity (2.29) can be rewritten in the form

$$d(-\tilde{F}_4 - F_2 \wedge B_2) = *j_5^{D4} \equiv *j_5^{Page}. \quad (2.31)$$

Note that the induced D4-brane current from the D6-branes in the presence of a B_2 field is canceled due to the D6-brane Bianchi identity⁸ $dF_2 = *j_7^{D6}$; however, in the presence

⁸In the presence of NS5-brane sources, there is an induced D4-brane current $A_1 \wedge *j_6^{NS5}$ on the NS5-brane sources. This is canceled against the term $A_1 \wedge H_3$ in \tilde{F}_4 due to the NS5-brane Bianchi identity $dH_3 = *j_6^{NS5}$.

| | Localized | Gauge-Invariant | Quantized | Conserved |
|----------------|-----------|-----------------|-----------|-----------|
| Brane Charge | Yes | Yes | | |
| Maxwell Charge | | Yes | | Yes |
| Page Charge | Yes | | Yes | Yes |

Table 1: The properties of different types of charges. The properties not marked by “Yes” hold only in some cases.

of gauge field flux on the D6-branes they do still contribute to the D4-brane Page charge. Equation (2.31) implies that the D4-brane Page charge, which is the integral of

$$-\hat{F}_4 \equiv -\tilde{F}_4 - F_2 \wedge B_2 = -d(A_3 + B_2 \wedge A_1), \quad (2.32)$$

will respect Gauss’ law and count the number of D4-branes, at the expense of not being gauge-invariant (under gauge transformations of the B_2 field). This expression for the integer quantized flux can also be inferred from the consideration of charge quantization in the D1-D3 system, reviewed in Appendix A.

The D4-brane Page charge in our example of the previous subsection is the flux of \hat{F}_4 on $\mathcal{M} \times S^2$, given by

$$\int_{\mathcal{M} \times S^2} (-\tilde{F}_4 - F_2 \wedge B_2) = 2\pi\alpha. \quad (2.33)$$

This turns out to be independent of r , as expected, due to a delicate cancelation. The Page charge is generally quantized, implying in our case the quantization condition

$$2\pi\alpha = (2\pi l_s)^3 g_s N_4. \quad (2.34)$$

Thus, we have identified the values of the undetermined parameters α and l in terms of the field theory data N_4 and b_∞ (we will review how b_∞ is related to the gauge couplings below).

To summarize, there are three distinct types of charges: Maxwell, Page, and brane. These charges have different characteristics with regards to the properties of localization, quantization, gauge invariance, and conservation, which we summarize in table 1. Localization refers to the fact that the sources are delta-function supported, and that the fluxes satisfy Gauss’ law. Quantization refers to the fact that charges are integers, in appropriate units.⁹ Gauge-invariance refers to invariance with respect to all gauge transformations. Conservation refers to the property that the total charge is invariant under dynamical evolution, such as time evolution or moving along a moduli space. In simple situations, all three notions of the charge coincide, so that all four properties can be satisfied simultaneously. In the presence of Chern-Simons terms and various background fluxes, however, the charges are distinct.

⁹Up to certain overall shifts which we will describe in more detail later.

The fact that the Page charge that we computed above is not gauge-invariant might raise concern as to whether this quantity can be given any physical meaning. Indeed, non-gauge-invariant quantities are always unphysical. In the case at hand, however, the gauge ambiguity of the Page charge is only due to large gauge transformations which shift B_2 by an element of the integer cohomology $H^2(\mathbf{Z})$. As long as we consider the Page charge modulo such a large shift, its value is physical. Since at infinity

$$-\frac{1}{(2\pi l_s)^3 g_s} \int_{\mathcal{M} \times S^2} F_2 \wedge B_2 = -\frac{k}{2} b_\infty, \quad (2.35)$$

the large gauge transformation $b \rightarrow b + 1$ causes N_4 to be ambiguous modulo $k/2$. Note that for odd k this suggests that the quantization of N_4 is in units of $1/2$; we will elaborate more on this issue later in this article. Ignoring the subtlety, for the time being, we learn that the ambiguity in N_4 is modulo discrete shifts. Note that these shifts also change the value of the D2-brane charge at the same time, as we will describe in more detail below.

The ambiguity in the Page charge has a beautiful physical interpretation. It means that the backgrounds dual to field theories which lead to different Page charges, related by the large gauge transformation, really have the same charges, so different field theories could lead to the same gravitational background, and one could have “cascading” RG flows between these theories (without needing any brane sources in the bulk). In the corresponding gravity solutions, the b field (and the Maxwell charge) changes continuously as a function of the radial coordinate. It is often most convenient to describe the solution in terms of $0 \leq b < 1$, and in order to do this in some range of the radial coordinate one sometimes needs to do large gauge transformations of b . These change the Page charge, and thus also the field theory which we use to describe the solution in the corresponding range of energies. Various dual descriptions have a window of energy scales where they are the most effective description of the system.¹⁰ In some cases, such as [13], these flows are related to Seiberg duality [19]; the interpretation of this via Page charges was described in [20]. In other cases [16, 17, 21, 22, 23, 24], like the solution we wrote above, the RG flows represent Higgsing of one gauge group to another.

Note that in the $3 + 1$ dimensional case, the field theories involved in cascades are not asymptotically free, so the cascade always had to be cut off somehow at large energies (large radial coordinates), otherwise it leads to gauge groups of arbitrarily large rank at high energies. In $2 + 1$ dimensions any gauge theory is asymptotically free, so a key feature of cascades in $2 + 1$ dimensions is that the gauge couplings are expected to asymptote to fixed values in the UV [25]. There are no obstructions such as “duality walls” which prevent one from flowing all the way to the UV [20]. This is what makes parameters such as b_∞ (related to the gauge couplings) and the Maxwell charges well-defined in our construction above.

¹⁰A similar structure appears in the context of Morita-equivalences of non-commutative field theory [9].

In the solution we wrote above, using the careful identification of the parameters α and l in terms of N_4 and b_∞ , we can write

$$b(r) = b_\infty V(r) - \frac{2N_4}{k}(1 - V(r)), \quad (2.36)$$

where $V(r \rightarrow \infty) = 1$ and $V(r \rightarrow 0) = 0$. This demonstrates explicitly how $b(r)$ interpolates from a continuous value b_∞ in the UV, to a discrete value $(-2N_4/k)$ in the IR.¹¹

2.3 D4-brane probes and the gauge couplings

One way to determine the relation between the string coupling g_s , b_∞ , and the gauge couplings g_{YM}^2 of the two groups of the field theory dual, as well as to determine the range of the effective window for a given duality frame in a cascade, is to analyze probe D4-branes and anti-D4-branes extended along $\mathbf{R}^{1,2}$ and wrapping Σ in this geometry. Recall that we expect to have a $U(N_2 + N_4) \times U(N_2)$ $\mathcal{N} = 4$ supersymmetric gauge theory, which arises by splitting the N_2 “regular D2-branes” into two types of “fractional D2-branes,” which can be thought of as a D4-brane and an anti-D4-brane wrapping Σ , such that the total D4-brane charge cancels, but such that the two branes together carry one unit of D2-brane charge (so, if one of them carries n units of D2-brane charge, the other should carry $(1 - n)$). Our original description involved D4-branes that carry no D2-brane charge, so it corresponds to $n = 0$. These gauge theories have a Coulomb branch corresponding to separating all these “fractional branes” along the directions transverse to the ALE space. On the gravity side, moving along this Coulomb branch is realized by moving around fractional branes in the background.

The general form of the D4-brane action is

$$S_{D4} = S_{DBI} + S_{WZ}, \quad (2.37)$$

with

$$S_{DBI} = -T_4 \int_{R^{1,2} \times \Sigma} d^5x e^{-\phi} \sqrt{-\det(G + F + B)}, \quad (2.38)$$

$$S_{WZ} = -T_4 \int_{R^{1,2} \times \Sigma} \left[\frac{1}{2} (F + B) \wedge (F + B) \wedge A_1 + (F + B) \wedge A_3 + A_5 \right], \quad (2.39)$$

and

$$T_p \equiv \frac{1}{g_s} (2\pi)^{-p} l_s^{-(p+1)}. \quad (2.40)$$

¹¹ Note that parameters similar to b_∞ , describing the positions of the D5-branes in the defect field theory, should exist in the complete description of M-theory on $TN_{k1} \times TN_{k2}$ mentioned in footnote 5 above, although these parameters are decoupled in the limit where one flows to the $2 + 1$ dimensional theory dual to $ALE_{k1} \times TN_{k2}$.

We decompose

$$S_{DBI} = -T_4 \int_{R^{1,2} \times \Sigma} d^5x e^{-\phi} \sqrt{-\det_{R^{1,2}}(G+B+F)} \sqrt{\det_{\Sigma}(G+B+F)}, \quad (2.41)$$

and proceed to extract the moduli space metric by allowing only the time derivatives of the position of the D4-brane in the space transverse to the D6-branes to be non-zero, in which case

$$\begin{aligned} G_{00} &= -H^{-1/2}V^{1/2}(1 + HV^{-1}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)), \\ G_{11} &= H^{-1/2}V^{1/2}, \quad G_{22} = H^{-1/2}V^{1/2}. \end{aligned} \quad (2.42)$$

We also take a vanishing worldvolume gauge field in the field theory directions, $F_{01} = F_{02} = F_{12} = 0$, so that

$$-e^{-\phi} \sqrt{-\det_{R^{1,2}}(G+B+F)} = -H^{-1} \left[1 - \frac{HV^{-1}}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + \dots \right]. \quad (2.43)$$

On the other hand, the contribution from the vanishing 2-cycle directions is

$$\int_{\Sigma} \sqrt{-\det_{\Sigma}(G+B+F)} = \frac{2}{kR_{11}} |2\pi^2 l_s^2 R_{11} kn \pm (lV + \alpha)|, \quad (2.44)$$

where \pm distinguishes a D4-brane from an anti-D4-brane, and n is the quantized magnetic flux of the worldvolume gauge field F through Σ (which is related to the D2-brane charge of the probe). The contribution from the Chern-Simons terms is

$$S_{WZ} = T_4 \int \frac{2}{kR_{11}} [2\pi^2 l_s^2 R_{11} kn \pm (lV + \alpha)] H^{-1} + \dots. \quad (2.45)$$

We can consider probes carrying different amounts of D2-brane charge; usually we will only look at the lightest D4-brane and anti-D4-brane, and view branes with other charges as bound states of these lightest branes with D2-branes.

Now, we see that

$$S = S_{DBI} + S_{WZ} \quad (2.46)$$

simplifies drastically provided

$$2\pi^2 l_s^2 R_{11} kn \pm (lV + \alpha) > 0. \quad (2.47)$$

In this case the terms with no derivatives cancel, and the relevant terms in the action have the form

$$S = T_4 \left(\left(2\pi^2 l_s^2 n \pm \frac{\alpha}{kR_{11}} \right) V^{-1} \pm \frac{l}{kR_{11}} \right) (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$$

$$\equiv \frac{2\pi^2 l_s^2 T_4}{g_{eff}^2(r)} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2), \quad (2.48)$$

where in the last line we defined a dimensionless effective coupling $g_{eff}^2(r)$ on the worldvolume (note that this coupling also appears in the kinetic term of the worldvolume gauge field). Substituting l and α in terms of b_∞ and N_4 , using the results (2.25) and (2.34) from the previous subsection, we have

$$\frac{1}{g_{eff}^2(r)} = (n \pm b_\infty) + \frac{R_{11}(nk \mp 2N_4)}{2r}. \quad (2.49)$$

Finally, we perform the standard map between gauge theory and dual string theory parameters

$$R_{11} = g_s l_s, \quad r = 2\pi l_s^2 \Phi, \quad g_s = g_{YM}^2 (2\pi)^{-(p-2)} l_s^{-(p-3)}, \quad (2.50)$$

where Φ is the vacuum expectation value of the scalar field in the $\mathcal{N} = 4$ gauge theory, to find

$$\frac{1}{g_{eff}^2(\Phi)} = (n \pm b_\infty) + \frac{g_{YM}^2 (nk \mp 2N_4)}{4\pi\Phi}. \quad (2.51)$$

This should be interpreted as the running coupling on the moduli space of the field theory, as we move along the Coulomb branch.

If one takes, for concreteness, the naive fractional brane values that we mentioned above, $n = 0$ for the “+” (D4-brane) and $n = 1$ for the “−” (anti-D4-brane), the effective gauge couplings take the form

$$\frac{1}{g_{eff1}^2(\Phi)} = b_\infty - \frac{g_{YM}^2 N_4}{2\pi\Phi}, \quad (2.52)$$

$$\frac{1}{g_{eff2}^2(\Phi)} = (1 - b_\infty) + \frac{g_{YM}^2 (k + 2N_4)}{4\pi\Phi}. \quad (2.53)$$

We expect these to be the inverse effective couplings on the moduli space of our $\mathcal{N} = 4$ $U(N_2 + N_4) \times U(N_2)$ gauge theory (up to an overall factor of g_{YM}^2), and this is indeed the case. Recall [26] that in $d = 3$ $\mathcal{N} = 4$ gauge theories, the effective gauge coupling as a function of position in the moduli space is one-loop exact. There is a constant term which is just the classical gauge coupling, and a one-loop correction, independent of the Yang-Mills coupling, proportional to the number of hypermultiplets in the fundamental representation N_f minus twice the number of colors N_c , divided by the radial position Φ on the Coulomb branch. This matches with the equations above, if we identify the gauge couplings of the two groups as $1/g_1^2 = b_\infty/g_{YM}^2$ and $1/g_2^2 = (1 - b_\infty)/g_{YM}^2$; these are precisely the expected couplings for D4-branes wrapping a vanishing cycle with a B_2 field proportional to b_∞ . The number of flavors of the $U(N_2 + N_4)$ group is $2N_2$ (from the two bifundamentals), so for this

group $N_f - 2N_c = -2N_4$, while the $U(N_2)$ gauge group has $2(N_2 + N_4) + k$ fundamental hypermultiplets, so it has $N_f - 2N_c = 2N_4 + k$. This agrees with the expressions above when we choose the k fundamental hypermultiplets to be charged under the $U(N_2)$ group.

As one moves the D4-branes along the Coulomb branch, there is a point,

$$\Phi = \frac{g_{YM}^2 N_4}{2\pi b_\infty}, \quad (2.54)$$

at which the effective coupling g_{eff1}^2 becomes infinite. This is an enhancon point [27], where the tension of the D4-brane wrapped on Σ is going to zero. Generally, in backgrounds like this, there are tensionless D4-branes sitting at the enhancon in the background [17, 21, 24], although they did not appear explicitly in our solution. If we try to move the D4-brane beyond this point, it no longer satisfies (2.47), so we are no longer on the moduli space. However, there are different probes that one can use beyond this point : we can use $n = 1$ for the “+” (D4-brane) probe and $n = 0$ for the “−” (anti-D4-brane) probe. For these probes, the effective couplings are

$$\frac{1}{g_{eff1}^2(\Phi)} = (1 + b_\infty) - \frac{g_{YM}^2(2N_4 - k)}{4\pi\Phi}, \quad (2.55)$$

$$\frac{1}{g_{eff2}^2(\Phi)} = (1 - (1 + b_\infty)) + \frac{g_{YM}^2 N_4}{2\pi\Phi}, \quad (2.56)$$

which are still positive in this region. Note that for $k = 0$ the change we made is the same as shifting b_∞ by one; the gauge transformation that shifts b_∞ by one also shifts n , and only their sum (for a D4-brane) is gauge-invariant. We will provide further interpretation of this shift in section 2.5 below, but for now, simply note that we can interpret the change in the effective action on the moduli space as related to a RG flow along a cascade (since, for theories with eight supercharges, the running couplings are the same as the effective couplings on the moduli space). This is best illustrated by drawing the inverse couplings on the two types of brane probes (chosen such that both probes have positive couplings) as a function of $g_{YM}^2 N_4 / 2\pi\Phi$ (see figure 1). Note that with each cascade transition, N_2 decreases (we will describe the precise change in more detail below). The reduced rank of the gauge group means that we have a smaller moduli space; the coordinates of the moduli space which “disappeared” correspond to the positions of the D4-branes sitting at the enhancon point (see [28] for details).

The cascade can end in the IR in one of two ways (see figure 1). One possibility is that we encounter an enhancon where all the branes sit (when this enhancon is at $\Phi > 0$, the solution inside the enhancon radius is just the solution with no fluxes). This is the only possibility when $k = 0$, and it also happens whenever no further transitions can be made (with positive rank gauge groups). In this case one expects the enhancon mechanism to capture the IR

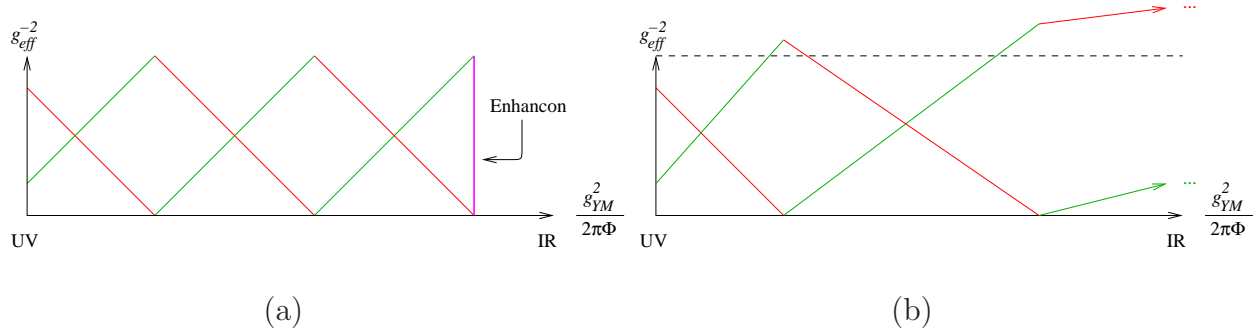


Figure 1: The effective gauge couplings of the two gauge groups on the moduli space, as a function of the moduli space coordinate; in theories with eight supercharges this may also be interpreted as the effective running coupling at a particular scale. Figure (a) represents the $k = 0$ case where the cascade terminates in the IR with an enhancecon. Figure (b) represents a cascade ending when $k > 2N_4$, so that the effective gauge couplings become small as $\Phi \rightarrow 0$ (and there may be a non-trivial SCFT there).

dynamics of the theory, namely the background should contain fractional branes sitting at the enhancecon carrying all the remaining charges. The second possibility, when $k > 0$, is that the value of N_4 could decrease until $N_4 < k/2$, and then both couplings start decreasing towards the IR so there are no further cascades.

2.4 The harmonic function and the decoupling limit

For the sake of completeness, let us briefly discuss the solution to the harmonic equation (2.22) and the decoupling limit. If one takes all of the D2, D4, and D6-branes to be coincident at the origin, (2.22) takes the form

$$-V \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) H(y, r) - \nabla_y^2 H(y, r) = \frac{l^2 V^4}{2r^4} \delta^4(\vec{y}) + (2\pi l_s)^5 g_s Q_2 \delta^4(\vec{y}) \delta^3(\vec{r}). \quad (2.57)$$

In taking the decoupling limit, one performs the standard scaling

$$r = \alpha' U, \quad y = \alpha' Y, \quad g_{YM2}^2 = g_s l_s^{-1}, \quad (2.58)$$

and one should also rescale

$$H(r, y) = \frac{1}{\alpha'^2} h(U, Y). \quad (2.59)$$

In this limit,

$$V^{-1} = 1 + \frac{g_{YM}^2 k}{2U} \quad (2.60)$$

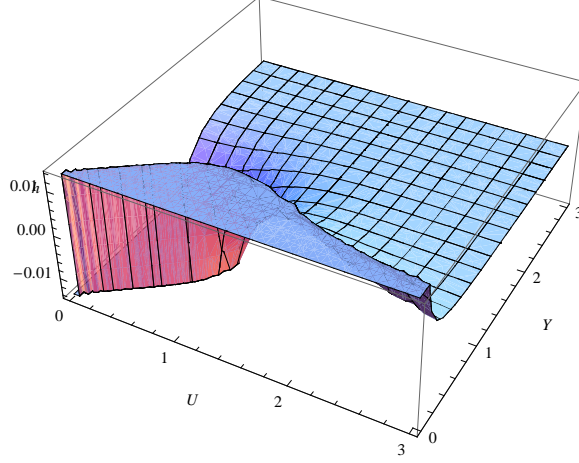


Figure 2: The solution to the harmonic equation in the decoupling limit, for $k = 0$.

is independent of α' . We also scale l and α according to (2.25) and (2.34). In terms of these variables, the harmonic equation takes the form

$$\begin{aligned} V \left(\frac{\partial^2}{\partial U^2} + \frac{2}{U} \frac{\partial}{\partial U} \right) h(Y, U) + \nabla_Y^2 h(Y, U) \\ = - \frac{(2\pi)^4 g_{YM2}^2 \left(N_4 + \frac{b_\infty k}{2} \right)^2 V^4}{2U^4} \delta^4(\vec{Y}) - (2\pi)^5 g_{YM2}^2 \left(N_2 + b_0 N_4 + \frac{N_6 b_0^2}{4} \right) \delta^4(\vec{Y}) \delta^3(\vec{U}), \end{aligned} \quad (2.61)$$

and is independent of α' . Just as in [14], all of the structures of (2.22) survive in this scaling limit. In terms of the solution to this warp factor equation, the supergravity background metric can be written in the standard form

$$\frac{ds_{IIA}^2}{\alpha'} = h^{-1/2}(-dt^2 + dx_1^2 + dx_2^2) + h^{1/2} dS_{ALE}^2 + h^{1/2}(dU^2 + U^2(d\theta^2 + \sin^2 \theta d\phi^2)), \quad (2.62)$$

where

$$ds_{ALE}^2 = \alpha'^2 dS_{ALE}^2 = \alpha'(dY^2 + Y^2 ds_{Lens}^2). \quad (2.63)$$

Although somewhat cumbersome, $h(U, Y)$ can be determined using various methods. As an example, we illustrate the form of $h(U, Y)$ for the case of $k = 0$ in figure 2.

2.5 Maxwell charges and the brane creation effect

Although we have identified the Page charge as the relevant quantity for imposing charge quantization conditions, Maxwell charges remain an important gauge-invariant parameter to distinguish between physically inequivalent configurations.

As reviewed above, Maxwell charges are defined as the flux of the gauge-invariant field strength at infinity. In many cases, the easiest way to compute the Maxwell charge is to

relate it to the flux defining the Page charge. In particular, the fluxes of

$$(-\hat{F}_4) = -\tilde{F}_4 - B_2 \wedge F_2, \quad (2.64)$$

$$\hat{F}_6 = *\tilde{F}_4 - B_2 \wedge (-\hat{F}_4) - \frac{1}{2}B_2 \wedge B_2 \wedge F_2, \quad (2.65)$$

define the Page charges of D4-branes and D2-branes, respectively, in the type IIA theory. This can then be used to compute the Maxwell charges where we write

$$Q_4^{Maxwell} = \frac{1}{(2\pi l_s)^3 g_s} \int_{\mathcal{M} \times S^2} (-\tilde{F}_4) = \frac{1}{(2\pi l_s)^3 g_s} \int_{\mathcal{M} \times S^2} [(-\hat{F}_4) + B_2 \wedge F_2], \quad (2.66)$$

$$\begin{aligned} Q_2^{Maxwell} &= \frac{1}{(2\pi l_s)^5 g_s} \int_{ALE \times S^2} *\tilde{F}_4 \\ &= \frac{1}{(2\pi l_s)^5 g_s} \int_{ALE \times S^2} \left[\hat{F}_6 + B_2 \wedge (-\hat{F}_4) + \frac{1}{2}B_2 \wedge B_2 \wedge F_2 \right]. \end{aligned} \quad (2.67)$$

Evaluating these integrals at infinity, and denoting the quantized Page charges by N_4 and N_2 as above, we obtain

$$Q_4^{Maxwell} = N_4 + \frac{1}{2}b_\infty N_6, \quad (2.68)$$

$$Q_2^{Maxwell} = N_2 + b_\infty N_4 + \frac{1}{4}b_\infty^2 N_6. \quad (2.69)$$

These gauge-invariant charges at infinity are well-defined in 2+1 dimensions. (In 3+1 dimensional cascading backgrounds, they grow logarithmically and are therefore infinite.)

There are two qualitatively distinct ways in which the value of b_∞ can change. One is to vary it continuously, keeping the Page charges N_2 , N_4 , and N_6 fixed. This is a physical deformation, and it changes the values of $Q_2^{Maxwell}$ and $Q_4^{Maxwell}$ accordingly. We refer to this operation as “sliding” b_∞ . Another possibility is to shift the value of b_∞ by one via a large gauge transformation. This also changes the values of N_2 and N_4 (as well as transforming the background fluxes), but since the gauge transformation is not a physical process, the gauge-invariant $Q_2^{Maxwell}$ and $Q_4^{Maxwell}$ are unchanged. We will refer to this procedure as “shifting” b_∞ . The “shift” can only vary b_∞ by an integer amount. It is clear from the equations above, that “shifting” $b_\infty \rightarrow b_\infty - 1$ takes $N_4 \rightarrow N_4 + \frac{1}{2}N_6$ and $N_2 \rightarrow N_2 + N_4 + \frac{1}{4}N_6$.

The behavior of Maxwell charges under combination of “slides” and “shifts” has interesting ramifications. It is useful to consider this process from the point of view of the brane configurations which reduce to the corresponding gauge theories at low energies [29, 30]. These brane configurations involve D3-branes stretched along a compact x^6 direction (as well as the three field theory directions), two NS5-branes stretching along the x^3, x^4, x^5 directions, and N_6 D5-branes stretching along the x^7, x^8, x^9 directions. Given some value of $0 < b_\infty < 1$, our claim is that the supergravity background (2.14)–(2.18) is dual to the field

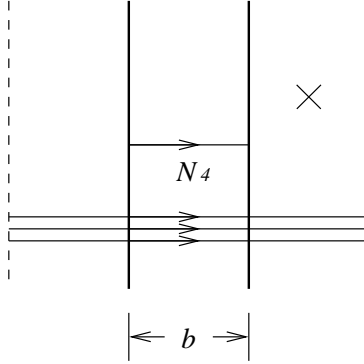


Figure 3: The brane configuration describing a $U(N_2) \times U(N_2 + N_4)$ theory with N_6 additional flavors charged under $U(N_2)$. The vertical solid lines represent the NS5-branes oriented along the 012345 directions. The horizontal solid lines represent the D3-branes extended along the 0126 directions. The \times represents the D5-brane extended along the 012789 directions. The vertical direction in the figure represents one of the 345 directions, whereas the horizontal direction in the figure represents the x^6 coordinate. The vertical dashed lines in the figure are identified as a result of the compactification along the x^6 direction. The N_4 fractional D3-branes stretch between the NS5-branes, whereas N_2 integer D3-branes stretch around the periodic direction. Similar conventions apply to all of our brane diagrams.

theory described by the brane diagram illustrated in figure 3, where b_∞ is the distance in x^6 between the two NS5-branes along the fractional branes (divided by the radius of the x^6 circle). For the moment, we ignore the vertical placement of the fractional branes and D5-branes in figure 3.

Suppose now we consider sliding b_∞ so that we let $b_\infty = 1 + c$ for some $0 < c < 1$. In the brane configuration we can realize this by moving one of the NS5-branes around the circle. Taking into account the brane creation effect when the NS5-branes cross the D5-branes, the brane configuration becomes the one on the left-hand side of figure 4, with N_6 D3-branes ending on the D5-branes. Now, taking advantage of the fact that moving the D5-branes in x^6 while preserving (locally) the linking number does not change the effective dynamics in 2+1 dimensions, we can move the D5-branes to obtain a different description of the same low-energy theory on the right of figure 4.¹²

The configuration on the right in figure 4 shows the D5-branes giving rise to flavors charged under the gauge group with larger rank, even though we started with the opposite situation in figure 3. We will elaborate on the exchange in gauge group with respect to

¹²It should be noted that as 3+1 dimensional defect theories, these two configurations are distinct, but the difference decouples as one flows from 3+1 to 2+1, essentially along the lines of the decoupling of b_∞ in flowing to the IR fixed point of the 2+1 dimensional theory.

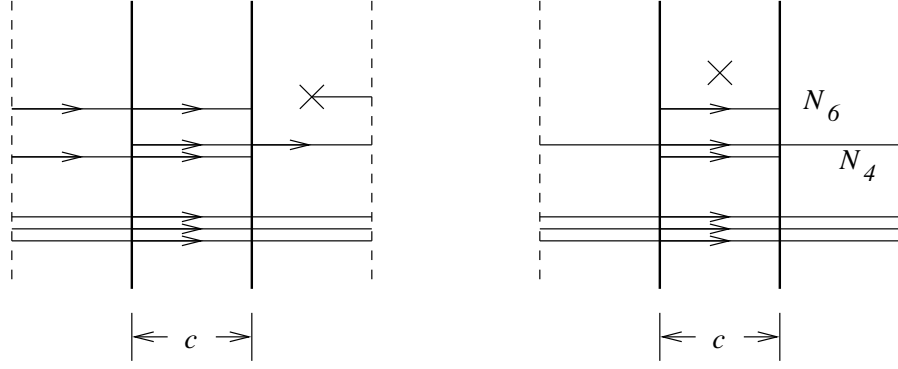


Figure 4: The same brane configuration after moving the NS5-brane on the right around the circle to the right (on the left), and (on the right) the same configuration after also moving the D5-branes to the right.

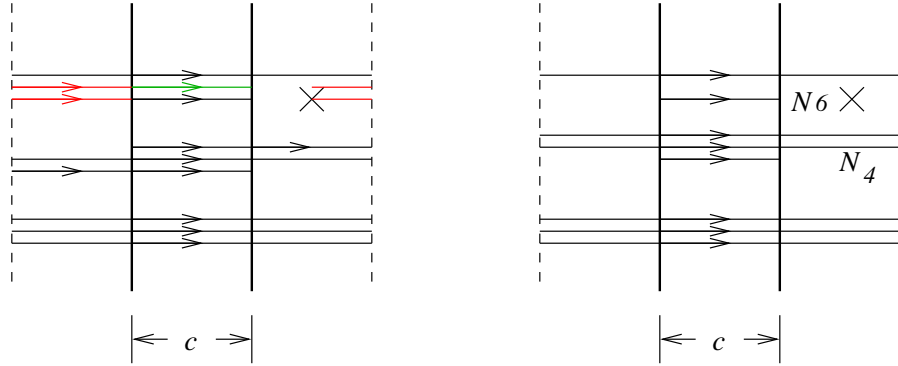


Figure 5: The brane configuration obtained after taking an NS5-brane twice around the circle (on the left), and (on the right) the same configuration after also moving the D5-branes.

which the flavors are charged in the next subsection. For now, simply note that sliding to $b_\infty = c + 2$ for $0 < c < 1$ will lead to the configuration illustrated in figure 5, where the flavor branes are back in their original position.¹³

The changes in the number of fractional branes, as we slide b_∞ , can be thought of as a consequence of a fundamental process where an NS5-brane sandwiches a D5-brane as they cross one another, as illustrated in figure 6.

After one performs a slide so that $b_\infty = c + 2$, it is useful to perform a “shift” to bring the value of b_∞ back to the value c , obeying $0 < c < 1$. This is just twice the transformation

¹³It might appear at first sight that the left figure violates the s -rule because two D3’s (in red) stretch from a D5 to the NS5. However, since these D3’s terminate on *different* NS5’s (in green) they do not violate the s -rule. We will discuss this issue further in the following section.

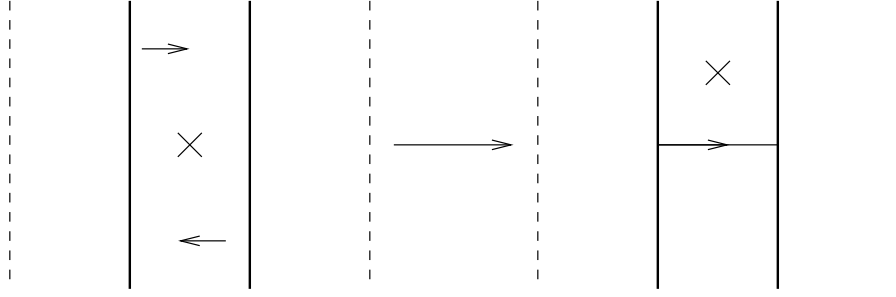


Figure 6: The “basic transition” that creates fractional branes when two NS5-branes cross each other in the presence of a D5-brane between them.

described earlier; since the “shift” keeps the Maxwell charges intact, one can read off the variation in the Page charges by rewriting

$$\begin{aligned}
 Q_2^{Maxwell} &= N_2 + (c+2)N_4 + \frac{1}{4}(c+2)^2N_6 \\
 &= (N_2 + 2N_4 + N_6) + c(N_4 + N_6) + \frac{1}{4}c^2N_6.
 \end{aligned}
 \tag{2.70}$$

Note that the change we obtain in the Page charges (which arises also from performing twice the single “shift+slide” transformation described above),

$$N_2 \rightarrow N_2 + 2N_4 + N_6, \tag{2.71}$$

$$N_4 \rightarrow N_4 + N_6, \tag{2.72}$$

precisely matches the change in the counting of integer and fractional branes illustrated in figure 5. We therefore learn that the precise forms of the Maxwell and Page charges carefully account for the brane creation effects [30]. In other words, one can view the brane creation effects as the logical consequence of the subtlety in the quantization of charges in the presence of Chern-Simons terms in the low-energy effective description of M-theory and type IIA supergravity. This will prove to be a powerful diagnostic tool.

2.6 Some generalizations

We close this section by discussing a few generalizations of the construction of the supergravity dual of $\mathcal{N} = 4$ theories in $2 + 1$ dimensions.

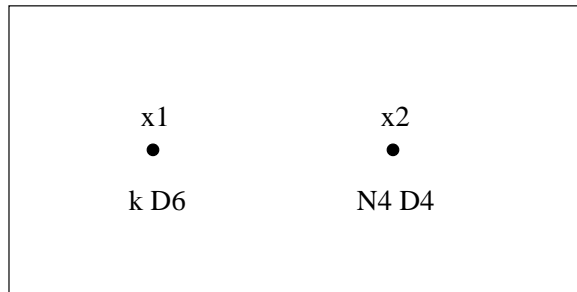


Figure 7: The positions in the transverse space ($x^7 \dots x^9$) of the D4-branes and the D6-branes.

2.6.1 Mass and moduli space deformation

One simple generalization is to consider D2-branes, D4-branes, and D6-branes placed at different points along the coordinates transverse to the D6-branes. Moving the D6-branes corresponds to changing the masses of the fundamental matter fields, and was considered briefly in [14]. Moving the D2-branes and the D4-branes corresponds to moving along the Coulomb branch of the moduli space (as we discussed for a single fractional brane in section 2.3). Here, for the sake of illustration, let us consider separating all of the N_4 D4-branes away from the k D6-branes, as illustrated in figure 7.

It is not too difficult to verify that

$$G_4^{SD} = (2\pi l_s)^2 R_{11} \omega \wedge (\eta + *_4 \eta), \quad (2.73)$$

for

$$\eta = - \left(\frac{1}{2} b_\infty k V - N_4 V \frac{R_{11} k}{2|\vec{x} - \vec{x}_2|} \right) d\Omega_1 - N_4 d\Omega_2 \quad (2.74)$$

(where $d\Omega_1$ is the volume form of a 2-sphere around the D6-branes, and $d\Omega_2$ is the volume form of a 2-sphere around the D4-branes), is self-dual, closed, and has the appropriate behavior at large distances (with appropriate sources). It should not be too difficult to construct an analogous expression for a general distribution of D4-branes and D6-branes in the three transverse directions. As in [14], solving for the harmonic function H is much more challenging in these general cases.

2.6.2 Flavors charged under $U(N_2)$ and $U(N_2 + N_4)$

Another generalization we can consider is changing the gauge group under which the fundamental hypermultiplets are charged. Recall that our claim is that the supergravity background (2.14)–(2.18) is dual to the field theory described by the brane configuration illus-

trated in figure 3, with the fundamental hypermultiplets charged under the lower rank group. What then would constitute the supergravity dual for the field theory described by the brane configuration illustrated in figure 4, with matter charged under the higher rank group ?

The answer to this question can be gleaned from the way in which we obtain the configuration of figure 4 in the first place. The configuration of figure 4 is obtained by “sliding” b_∞ by one, and then “shifting” it by -1 (if we want to bring it back to $[0, 1]$). The brane configuration suggests that we have a $U(N'_2) \times U(N'_2 + N'_4)$ theory with

$$N'_2 = N_2 + N_4, \quad N'_4 = N_4 + N_6. \quad (2.75)$$

The Maxwell charge is the same as our original expression (2.68),(2.69) for the charges when $b = b_\infty + 1$ (since it is not changed by the “shift”); expressing it in terms of the new charges we obtain

$$Q_2^{Maxwell} = \left(N'_2 + \frac{N_6}{4}\right) + \left(N'_4 - \frac{N_6}{2}\right)b_\infty + \frac{N_6}{4}b_\infty^2, \quad (2.76)$$

$$Q_4^{Maxwell} = \left(N'_4 - \frac{N_6}{2}\right) + \frac{N_6}{2}b_\infty. \quad (2.77)$$

In terms of the new charges, the large gauge transformation has shifted the Page charges to

$$Q_2^{Page} = N'_2 + \frac{1}{4}N_6, \quad Q_4^{Page} = N'_4 - \frac{1}{2}N_6. \quad (2.78)$$

The fact that this expression for the Page charges is different from our previous expression is consistent with the fact that we are describing a different configuration now, with the fundamental fields charged under a different group. From the point of view of the type IIA picture, a simple way to understand the charges (2.78) is to say that we now have a self-dual gauge field flux on the worldvolume of each of the D6-branes (using the normalization where $\mathcal{F} = B + 2\pi\alpha'F$),

$$F = 2\pi\omega_2. \quad (2.79)$$

Namely, we identify the D6-brane without the flux as giving a flavor for one gauge group, and the D6-brane with this flux as giving a flavor for the other gauge group. A curious fact here is that these self-dual fluxes are quantized such that the induced D4-brane charge on the D6-brane worldvolume is half-integral.

The problem with using the slide and shift of b_∞ by one is that one can only move all k D5-branes in the type IIB description at once under this transformation. What would it mean to move some of the D5-branes, so that there are N_{6-} flavors charged with respect to the $U(N_2 + N_4)$ gauge group and N_{6+} flavors charged with respect to the $U(N_2)$ group, as illustrated in figure 8? If we take the D6-branes in the type IIA description as probes, the

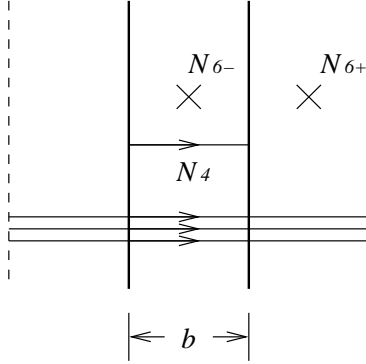


Figure 8: A brane configuration with flavors charged under both gauge groups.

answer, based on the discussion above, is that we need to turn on a self-dual flux for the $U(1)$ gauge field on the N_{6-} D6-branes. This picture is consistent with what is described in [16, 20] in related contexts.

As far as charges are concerned, we can write down expressions consistent with our previous expressions by taking

$$Q_2^{Maxwell} = \left(N_2 + \frac{N_{6-}}{4}\right) + \left(N_4 - \frac{N_{6-}}{2}\right)b_\infty + \frac{N_6}{4}b_\infty^2, \quad (2.80)$$

$$Q_4^{Maxwell} = \left(N_4 - \frac{N_{6-}}{2}\right) + \frac{N_6}{2}b_\infty, \quad (2.81)$$

and

$$Q_2^{Page} = N_2 + \frac{1}{4}N_{6-}, \quad Q_4^{Page} = N_4 - \frac{1}{2}N_{6-}, \quad (2.82)$$

where we have dropped the primes from the N_2 's and the N_4 's. These Page charges can then be used to construct the supergravity duals to these theories using (2.14)–(2.18) and (2.34) and (2.25), and using the appropriate Page charges to specify the discrete adjustable parameters of the solution.

There is one issue to keep in mind: while there are 4 sets of discrete data, N_2 , N_4 , N_{6+} , and N_{6-} , in the field theory, there are only three Page charges Q_2^{Page} , Q_4^{Page} , and Q_6^{Page} which determine our gravity solution. Thus, there is a degeneracy in the values of the Page charges corresponding to distinct field theories. This degeneracy is most likely lifted by taking into account the fluxes on the D6-branes; in similar cases these can be thought of as Wilson lines on the D6-branes [31, 32, 33], which, when lifted to M theory, become topological data of the 3-forms [34]. While there are sufficient degrees of freedom in the topological data of the 3-forms to lift the degeneracy between the field theory and gravity data, identifying the one-to-one map between gauge equivalence classes of gravity backgrounds and distinct field

theories seems rather challenging. This issue is closely related to that of mapping out the moduli space of the field theory from the supergravity point of view, which remains generally a subtle issue. We leave detailed consideration of these issues for future work.

3 Flux quantization in the ABJM and ABJ models

In this section, we repeat the analysis of quantization of charges and fluxes for the ABJ and ABJM systems (with Yang-Mills kinetic terms), namely for $U(N_2)_k \times U(N_2 + N_4)_{-k}$ Yang-Mills-Chern-Simons theories with $\mathcal{N} = 3$ supersymmetry in three dimensions (for simplicity, in this case we do not add any matter fields to those already present in the ABJM construction). Our goal is to demonstrate the consistency of having a continuous value of b_∞ but a discrete value for b in the IR, and to understand better how the gauge potential flux b is quantized as a consequence of charge quantization. We will reverse the order of presentation of the previous section, starting by looking at the counting of branes in the corresponding brane configuration, and inferring from this an expected form for the Maxwell and Page charges. Although we will not explicitly construct the dual supergravity solution here, we will have enough information from charge quantization to reproduce similar conclusions to those of section 2. We will then show that the quantization of b thus obtained is consistent with a variety of checks.

3.1 Brane configurations and Maxwell charges

The $\mathcal{N} = 3$ Yang-Mills-Chern-Simons theories arise as the low-energy effective field theory of D3-branes stretched on a circle, intersecting one NS5-brane and one $(1, k)$ 5-brane in type IIB string theory. Models of this type were originally considered in [35, 36]. Following the convention of [5], we take the D3-branes to be oriented along the coordinates 0126 (with x^6 on a circle), the NS5-brane to be oriented along 012345, and the $(1, k)$ 5-brane to be oriented along $012[3, 7]_\theta[4, 8]_\theta[5, 9]_\theta$ with $\tan(\theta) = 1/g_s k$ (assuming that the RR axion vanishes). We allow some number N_2 of “regular branes” which are D3-branes going all the way around the circle, and some number N_4 of “fractional branes,” which are D3-branes stretched just along one segment of the circle. Such a brane configuration is illustrated in figure 9. As in the previous section, in the brane configuration we have a parameter b_∞ which is the distance between the pair of 5-branes, and which is classically related to the ratio between the gauge couplings. In the zero slope limit, this system becomes a $3+1$ dimensional Yang-Mills theory on $\mathbf{R}^{1,2} \times S^1$, with a pair of domain wall defects localized on the S^1 . We will discuss here the further limit where we decouple the $3+1$ dimensional dynamics and flow to a $2+1$ dimensional theory, by taking the radius of the circle in the type IIB description to be small.

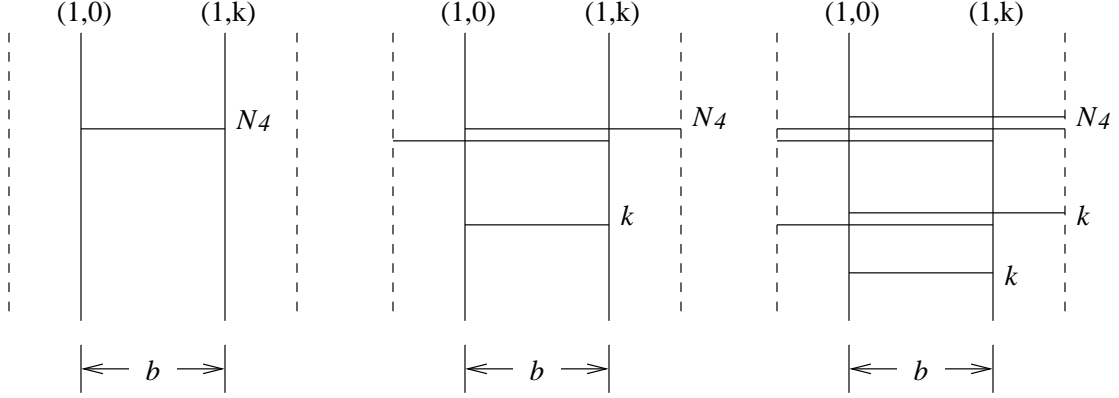


Figure 9: The brane configuration for the $\mathcal{N} = 3$ theories, and its change upon “sliding” $b \rightarrow b + 1 \rightarrow b + 2$. In the $\mathcal{N} = 3$ theories, fractional branes are *not* free to move in the vertical direction. We will nonetheless separate the branes in the vertical directions to avoid cluttering the figure. The N_2 integer branes, winding all the way around the periodic direction, have also been suppressed in the figure.

As in the previous section, one can follow the brane creation effects by sliding b_∞ past integer values. Each time the NS5-brane crosses the $(1, k)$ 5-brane, k D3-branes are created [30] (see figure 9). Thus, for each slide by a full cycle, the number of integer and fractional branes appears to shift according to

$$N_2 \rightarrow N_2 + N_4, \quad N_4 \rightarrow N_4 + k. \quad (3.1)$$

In this setup, it is not necessary to shift b by 2 to return to the same configuration, since there are no fundamental matter fields, and so there is no issue of keeping track of the gauge group with respect to which they are charged.

One subtle issue here is that the brane configurations that we obtain by these slides of b_∞ at first sight seem to violate the s -rule, which says that there can be at most k D3-branes stretched between an NS5-brane and a $(1, k)$ 5-brane [30, 35, 36]. The original configuration on the left of figure 9 is compatible with the s -rule if $0 \leq N_4 \leq k$, but the other configurations seem to have more than k “fractional D3-branes” stretched between the NS5-brane and the $(1, k)$ 5-brane. However, when the branes live on a circle, we have to be careful about applying the s -rule. In the case that we have a fractional brane together with a regular brane, we could interpret this either as one D3-brane stretched directly along the segment from the NS5-brane to the $(1, k)$ 5-brane, and another D3-brane wrapping the circle, or as a single D3-brane that winds around the circle more than once. In other words, in the covering space of the circle, we can have D3-branes that stretch between the NS5-brane and different images of the $(1, k)$ 5-brane [37], and the “modified s -rule” just tells us that there can be

at most k D3-branes stretched between the NS5-brane and a specific image of the $(1, k)$ 5-brane. Such a modified s -rule is necessary for the low-energy physics to be independent of x^6 positions of branes, even in highly supersymmetric configurations such as those of the previous section. It also follows from the derivation [38, 39] of the s -rule (in a U-dual frame) by considering strings between two types of D-branes (say, a D0-brane and a D8-brane), showing that they have a single fermionic ground state, and noting that in supersymmetric configurations this ground state is the only open string that can be excited, and that this single fermionic ground state can be excited (in a supersymmetric configuration) either zero times or once (and the brane creation effect goes from one of these states to the other). When we have a circle, we have different open string fermionic ground states, that stretch between the two branes while also going n times around the circle, and each of these open string modes can be excited either zero times or one time; each shift of $b_\infty \rightarrow b_\infty + 1$ excites one of these modes, but the modes excited by different shifts have different winding numbers, so they can be independently excited, and there is no violation of the s -rule. The fact that all the brane configurations of figure 9 seem to have supersymmetric vacua has interesting implications, which we will discuss in the next section.

In any case, the pattern (3.1) of changes in N_2 and N_4 implies that the Maxwell charges for this configuration must take the form

$$Q_2^{Maxwell} = N_2 + \left(N_4 - \frac{k}{2}\right) b_\infty + \frac{k}{2} b_\infty^2, \quad (3.2)$$

$$Q_4^{Maxwell} = \left(N_4 - \frac{k}{2}\right) + k b_\infty, \quad (3.3)$$

(up to a possible overall shift of Q_2 by a multiple of k which is independent of b_∞), as can be seen by substituting $b_\infty = c + 1$ and expanding according to the powers of c :

$$Q_2^{Maxwell} = (N_2 + N_4) + \left((N_4 + k) - \frac{k}{2}\right) c + \frac{k}{2} c^2, \quad (3.4)$$

$$Q_4^{Maxwell} = \left((N_4 + k) - \frac{k}{2}\right) + k c. \quad (3.5)$$

Note that (up to terms which depend only on k to which our analysis above is not sensitive), this is equivalent to (2.80) and (2.81) upon substituting

$$N_6 = 2k, \quad N_{6+} = N_{6-} = k. \quad (3.6)$$

A discussion similar to our discussion around equations (2.68), (2.69), then implies that the Page charges for this system are shifted as (up to a possible shift of Q_2 by a multiple of k)

$$Q_2^{Page} = N_2, \quad (3.7)$$

$$Q_4^{Page} = N_4 - \frac{k}{2}. \quad (3.8)$$

In the rest of this section, we will provide evidence supporting this expectation from the perspective of charge and flux quantization in supergravity.

3.2 Charge and flux quantization in supergravity duals of Yang-Mills-Chern-Simons-Matter theories

In this section, we examine the issues of charges and topology in the dual gravitational description of the Yang-Mills-Chern-Simons-Matter theory which arises as the zero slope limit of the brane configurations described in the previous subsection.

The starting point is the metric of Lee, Weinberg, and Yi [40] which describes the M-theory lift of overlapping 5-branes [41], namely, the configuration of the previous subsection without the D3-branes. In the case of two overlapping 5-branes with D5-brane and NS5-brane charges (p_1, p_2) and $(\tilde{p}_1, \tilde{p}_2)$, it is given by

$$ds_{LWY}^2 = V_{ij} d\vec{x}_i d\vec{x}_j + (V^{-1})^{ij} R_i R_j (d\varphi_i + A_i)(d\varphi_j + A_j), \quad (3.9)$$

where

$$V_{ij} = \delta_{ij} + \frac{1}{2} \frac{R_i p_i R_j p_j}{|R_1 p_1 \vec{x}_1 + R_2 p_2 \vec{x}_2|} + \frac{1}{2} \frac{R_i \tilde{p}_i R_j \tilde{p}_j}{|R_1 \tilde{p}_1 \vec{x}_1 + R_2 \tilde{p}_2 \vec{x}_2|}, \quad (3.10)$$

$(i, j = 1, 2)$, \vec{x}_1, \vec{x}_2 are two 3-vectors, and $\varphi_i \equiv \varphi_i + 2\pi$. A_1 and A_2 are two one-forms whose form we will not need explicitly. For the theories we are interested in, we have as in the previous subsection

$$(p_1, p_2) = (1, 0), \quad (\tilde{p}_1, \tilde{p}_2) = (1, k). \quad (3.11)$$

R_1 and R_2 are radii of $S^1 \times S^1$, which we identify as the 6 and 11 directions, respectively. Therefore, they are related to the field theory parameters by

$$R_1 = \frac{2\pi\alpha'}{L}, \quad R_2 = R_{11} = g_s l_s = g_{YM2}^2 \alpha', \quad (3.12)$$

where L is the radius of the type IIB circle that the D3-branes live on. In order to decouple the excited string states and the $3 + 1$ dimensional dynamical degrees of freedom, and flow to a Yang-Mills-Chern-Simons-Matter system in $2 + 1$ dimensions, we must scale $\alpha', L \rightarrow 0$, which means taking $R_2 \rightarrow 0, R_2/R_1 \rightarrow 0$.

In the region near the core, this LWY geometry asymptotes to a simple $\mathbf{R}^8/\mathbf{Z}_k$ geometry [5]. This is the regime where the geometry captures the dynamics of the $\mathcal{N} = 6, 8$ superconformal fixed point that the $2 + 1$ dimensional gauge theory flows to in the IR.

Now we can consider repeating the analysis of the previous section, replacing $ALE \times TN_k$ with the LWY space. The strategy is to add N_2 M2-branes and N_4 M5-branes wrapped on the torsion 3-cycle of $\mathbf{R}^8/\mathbf{Z}_k$, and to explicitly construct the (anti-)self-dual 4-form on the

LWY geometry with the appropriate boundary condition, as well as to solve for the harmonic function in order to determine the warp factor. Unfortunately, despite the fact that LWY, being a hyper Kähler manifold, has quite a bit of structure, it is a far more complicated space than the $ALE \times TN_k$ geometry (see e.g. [42] for the discussion of the harmonic function on this space). Fortunately, for the purpose of understanding the charges and their quantization, it is sufficient to understand the broad topological structure of these spaces. We leave the interesting mathematical exercise of finding the (anti-)self-dual 4-form on LWY and the precise solutions dual to the Yang-Mills-Chern-Simons theories for future work.

In order to better understand the topology of the LWY space, let us first focus on the region near the origin. There, we find $\mathbf{R}^8/\mathbf{Z}_k$ which is a cone whose base is S^7/\mathbf{Z}_k . We will assume in the rest of this section that the geometry in the IR when we add the additional branes is the $AdS_4 \times S^7/\mathbf{Z}_k$ geometry of [5, 6]; this is expected to be true at least for $0 \leq N_4 \leq k$, and in the next section we will suggest that it may have a wider range of validity. Reducing S^7 on its Hopf fiber gives rise to a \mathbf{CP}^3 manifold, whose Betti numbers are $(1, 0, 1, 0, 1, 0, 1)$. Because \mathbf{CP}^3 is also a Kähler manifold, there is a Kähler form J , and its powers are dual to the homology cycles on \mathbf{CP}^3 , namely

$$\begin{aligned} J &\leftrightarrow \mathbf{CP}^1, \\ J \wedge J &\leftrightarrow \mathbf{CP}^2, \\ J \wedge J \wedge J &\leftrightarrow \mathbf{CP}^3. \end{aligned} \tag{3.13}$$

These forms are normalized such that

$$\int_{\mathbf{CP}^1} J = 1, \quad \int_{\mathbf{CP}^2} J \wedge J = 1, \quad \int_{\mathbf{CP}^3} J \wedge J \wedge J = 1. \tag{3.14}$$

The RR 2-form field strength which one finds upon reducing M-theory on LWY to type IIA string theory along the Hopf fiber is proportional to J :

$$F_2 = 2\pi k R_{11} J. \tag{3.15}$$

In the type IIA IR geometry, there is a \mathbf{CP}^3 at each radial position, giving a foliation of the geometry transverse to the D2-branes. When this geometry is embedded inside the LWY geometry, one expects an analogous foliation to continue into the UV, with a corresponding two-form J . Let us therefore assume that the topological structure of \mathbf{CP}^3 , its homology cycles and their relation to the J 's, persist throughout the geometry.

In the type IIA language, the only 2-cycle on which one could have an NS-NS 2-form potential associated with the distance b_∞ is the \mathbf{CP}^1 . This suggests that the NS-NS 2-form potential should interpolate between

$$B = (2\pi)^2 \alpha' b_\infty J \tag{3.16}$$

in the UV region of the geometry, and

$$B = (2\pi)^2 \alpha' b_0 J \quad (3.17)$$

for some discrete b_0 (related to N_4 [6]) in the IR.

We now have most of the ingredients necessary to explore the issues of charge quantization. Let us first consider the charge of D4-branes wrapped on \mathbf{CP}^1 (which are the type IIA image of the “fractional branes” discussed above). The Page charge of these branes, as in the previous section, is given by the flux of \hat{F}_4 through the dual cycle \mathbf{CP}^2 . One important feature of the Page charge is the fact that Gauss’ law holds. So, computing this flux should give the same result in the IR and in the UV. To compute the Page charge in the IR, one evaluates

$$Q_4^{Page} = \frac{1}{(2\pi l_s)^3 g_s} \int_{\mathbf{CP}^2} (-\hat{F}_4) = \frac{1}{(2\pi l_s)^3 g_s} \int_{\mathbf{CP}^2} (-\tilde{F}_4 - B_2 \wedge F_2). \quad (3.18)$$

In the IR, however, we do not expect \tilde{F}_4 to contribute (since it vanishes in the $AdS_4 \times S^7/\mathbf{Z}_k$ background), so we have

$$Q_4^{Page} = -\frac{1}{(2\pi l_s)^3 g_s} \int_{\mathbf{CP}^2} B_2 \wedge F_2 = -k b_0. \quad (3.19)$$

Thus, we learn that [6]

$$b_0 = -\frac{Q_4^{Page}}{k} \quad (3.20)$$

is quantized in units of $1/k$, since the Page charge Q_4^{Page} is expected to be quantized.

Once we know the Page charge, it is straightforward to compute the Maxwell charges in the UV :

$$Q_2^{Maxwell} = Q_2^{Page} + b_\infty Q_4^{Page} + \frac{k}{2} b_\infty^2, \quad (3.21)$$

$$Q_4^{Maxwell} = Q_4^{Page} + k b_\infty, \quad (3.22)$$

using (2.67), (2.66), and (3.16). Note that these results are in complete agreement with (3.2) and (3.3), provided we identify the Page charges according to (3.7) and (3.8), in which Q_4^{Page} is shifted with respect to the naive integer quantized value by $k/2$. Let us now explain this shift.

3.3 Freed-Witten anomaly and a half integral shift in torsion flux

In sections 3.1 and 3.2, we saw that the quantized charges in the dual gravitational description of the Yang-Mills-Chern-Simons-Matter system were compatible with the expectation based on brane creation effects, provided that the charges (3.7) and (3.8) were quantized with a

k -dependent fractional shift. Since the shift is in the Page charge which satisfies Gauss' law, it suffices to understand this shift in the IR (ABJM) limit.

In section 2.6.2 we also encountered a half-integral shift in the D4-brane charge, which was due to the possibility of turning on an integral self-dual flux on the worldvolume of D6-branes, leading to a half-integral D4-brane charge. Here, there are no D6-branes in isolation, so the same argument can not be applied.

We can still try to use arguments similar to those of the previous section by isolating a D6-brane. To do this, consider a probe D6-brane wrapped on the \mathbf{CP}^2 cycle and extended along the field theory directions. Such a brane cannot be static, but this is not relevant for our considerations; it will behave like a domain wall in AdS_4 , separating a region with $(k - 1)$ units of D6-brane charge (F_2 flux) and a region with k units.

Because the manifold \mathbf{CP}^2 is not spin, the D-brane wrapped on it is subject to the Freed-Witten anomaly [11]. This anomaly arises because the path integral measure for the fermions on such a D-brane is not globally well-defined by itself. However, the total path integral measure is well-defined if the worldvolume gauge field carries half-integral flux through the \mathbf{CP}^1 homology 2-cycle of the \mathbf{CP}^2 . This flux induces (if there is no B_2 field) a $1/2$ -unit of D4-brane Page charge and a $1/8$ -unit of D2-brane Page charge on the D6-brane, due to the Chern-Simons coupling $C \wedge e^{B+F}$ on its worldvolume. This implies that when we go from the background with k units of F_2 flux to the background with $(k - 1)$ units of F_2 flux by crossing the domain wall, the D4-brane Page charge shifts from an integer to a half-integer or vice versa. Since for $k = 0$ we expect the charge to be an integer (in the brane configuration; of course in this case there is no $AdS_4 \times S^7/\mathbf{Z}_k$ region), this implies that the fractional part of Q_4^{Page} should be equal to that of $k/2$, which is consistent with equation (3.8).

While this result is in gratifying agreement with the analysis based on brane configurations, it has one important consequence for the dualities of ABJ and ABJM. This is because charge quantization now imposes a condition (3.20) on b_0 which includes a shift by $1/2$:

$$b_0 = -\frac{N_4}{k} + \frac{1}{2}. \quad (3.23)$$

We claim that the brane creation effects, as well as the careful consideration of charge quantizations, suggest this shift in the value of b_0 by one half, compared to the naive value assumed in [5, 6].¹⁴ Note in particular that even for the $U(N) \times U(N)$ theory of [5], this implies that the B_2 field is non-vanishing.

The same shift occurs also in the full solutions describing $\mathcal{N} = 3$ YMCS theories flowing

¹⁴A similar shift by a half can also be seen from the M theory perspective [43], but we will not discuss this here.

to the $\mathcal{N} = 6$ SCFTs. One class of solutions of this type was constructed in [42]. In that paper it was assumed that $b = 0$, but it is clear that the same solutions exist with an additional B_2 -field of the form $b = -N_4/k + 1/2$ for $N_4 = 0, \dots, k-1$. Since the B_2 -field is constant, these solutions have $b_0 = b_\infty$; from the point of view of the discussion in this section, this means that they have a specific relation between the value of N_4 and the ratio between the two gauge couplings in the UV. In particular, for the case of $b = 1/2$ these solutions describe the $U(N) \times U(N)$ theories with equal gauge couplings in the UV, which gives a parity-invariant theory. Generically, of course, there will be no such relation between the UV gauge couplings and N_4 ; it would be interesting to construct solutions for this more general case.¹⁵

3.4 Some consistency tests

In [5, 6] several tests of the conjectured correspondence between $\mathcal{N} = 6$ Chern-Simons-matter theories and type IIA string theory on $AdS_4 \times \mathbf{CP}^3$ *without* the shift in b_0 by $1/2$ were presented. In this section, we will show how each of these arguments remains valid also after the shift of b_0 by $1/2$. Some of these tests can be regarded as independent arguments for why b_0 should be shifted by $1/2$ (which use the Freed-Witten anomaly, but do not directly use the shifted Page charge).

3.4.1 Domain wall brane charge

In section 3.3, the D6-brane wrapped on \mathbf{CP}^2 played an important role. Let us consider the D2-brane “brane charge” associated with this object.

First, we need to account for the $1/8$ units of D2-brane charge which were induced on the D6-brane, contributing to the Page charge. This modifies (3.2) in the ABJM limit to read

$$Q_2^{Maxwell} = \left(N_2 + \frac{k}{8}\right) + b_0 \left(N_4 - \frac{k}{2}\right) + \frac{k}{2}b_0^2 + Q_2^{anomaly} \quad (3.24)$$

where¹⁶

$$b_0 = -\frac{N_4}{k} + \frac{1}{2}. \quad (3.25)$$

We have also introduced the contribution due to higher curvature terms

$$Q_2^{anomaly} = -\frac{1}{24} \left(k - \frac{1}{k}\right) \quad (3.26)$$

¹⁵If we vary b_∞ keeping N_4 fixed, there will be additional gravitational back reaction which we discuss briefly in section 4.

¹⁶For $k = 2$ and $N_4 = 0, 1$, this evaluates to $Q_2^{Maxwell} = N_2 - 1/16$ and $Q_2^{Maxwell} = N_2 + 3/16$, in agreement with the computation of [44].

which was computed in [12]. These corrections, depending only on k , do not affect the brane manipulation argument from the previous section. Nonetheless, they turn out to be critical in some of the charge quantization issues as was the case in [11].

In particular, the Page charge, which is related to the Maxwell charge (2.67) as

$$\begin{aligned} Q_2^{Page} &= \frac{1}{(2\pi l_s)^5 g_s} \int_{\mathbf{CP}^3} \hat{F}_6 \\ &= \frac{1}{(2\pi l_s)^5 g_s} \int_{\mathbf{CP}^3} \left[* \tilde{F}_4 - B_2 \wedge (-\hat{F}_4) - \frac{1}{2} B_2 \wedge B_2 \wedge F_2 \right], \end{aligned} \quad (3.27)$$

is now shifted to take on a non-integer value (at leading order in large k)

$$Q_2^{Page} = N_2 + \frac{k}{8} - \frac{k}{24} = N_2 + \frac{k}{12}. \quad (3.28)$$

The D2-brane brane charge on the domain wall of section 3.3, which shifts the value of k by one, can readily be computed:

$$Q_2^{Brane} = Q_2^{Maxwell}(N_2, N_4, k) - Q_2^{Maxwell}(N_2, N_4, k-1) = \frac{1}{2} \left(\frac{N_4}{k} \right)^2 - \frac{1}{24} + \mathcal{O}(k^{-1}), \quad (3.29)$$

where we drop terms subleading in $1/k$ for the purpose of comparing this result with the probe approximation. This is indeed in agreement with the expected contributions from the terms in the worldvolume action [45]

$$S_6^R \sim \int C \wedge e^{F+B} \wedge \sqrt{\frac{\hat{\mathcal{A}}(4\pi^2 R_T)}{\hat{\mathcal{A}}(4\pi^2 R_N)}}. \quad (3.30)$$

Here,

$$\hat{\mathcal{A}} = 1 - \frac{1}{24} p_1 + \frac{1}{5760} (7p_1^2 - 4p_2) \quad (3.31)$$

is the ‘‘A-roof’’ genus, which is expressed in terms of Pontryagin classes

$$p_1 = -\frac{1}{2(2\pi)^2} \text{Tr} R^2, \quad p_2 = \frac{1}{8(2\pi)^4} [(\text{Tr} R^2)^2 - 2\text{Tr} R^4]. \quad (3.32)$$

It is useful to note that for \mathbf{CP}^2 embedded into \mathbf{CP}^3 ,

$$\int_{\mathbf{CP}^2} p_1(T) = 3, \quad (3.33)$$

$$\int_{\mathbf{CP}^2} p_1(N) = 1, \quad (3.34)$$

are the first Pontryagin classes for the tangent and the normal bundles, respectively [46].

The resulting D2-brane charge is

$$Q_2^{Brane} = Q_2^{WZ} + Q_2^R, \quad (3.35)$$

where (recalling that the D6-brane has a half-integer flux of its gauge field F)

$$Q_2^{WZ} = \frac{1}{(2\pi l_s)^4} \int_{\mathbf{CP}^2} \left[\frac{1}{2} (B_2 + F) \wedge (B_2 + F) \right] = \frac{1}{2} \left(\frac{N_4}{k} \right)^2, \quad (3.36)$$

and the higher curvature terms

$$Q_2^R = \int_{\mathbf{CP}^2} \frac{1}{48} (-p_1(T) + p_1(N)) = -\frac{1}{24}, \quad (3.37)$$

giving rise to a result matching (3.29).¹⁷

3.4.2 Di-baryon vertex

A D4-brane wrapping the \mathbf{CP}^2 cycle in $AdS_4 \times \mathbf{CP}^3$ describes a particle-like object in AdS_4 . The Ramond-Ramond fluxes induce an electric field on the D4-brane worldvolume, which must be canceled by some number of fundamental strings ending on the D4-brane, as for the baryon vertex in $AdS_5 \times S^5$ [47]. The number of such strings follows from the CS couplings on the D4-brane worldvolume, which are proportional to

$$\begin{aligned} S^{WZ} &\sim - \int_{\mathbf{R} \times \mathbf{CP}^2} A \wedge (F_4 + (B_2 + F) \wedge F_2) \\ &= - \int_{\mathbf{R} \times \mathbf{CP}^2} A \wedge (\hat{F}_4 + F \wedge F_2). \end{aligned} \quad (3.38)$$

This object was identified in [5,6] with a di-baryon, made by contracting N_2 bi-fundamentals to a singlet using the epsilon symbol in $SU(N_2)$; in the $U(N_2) \times U(N_2)$ theory such an object can be a singlet also of the other $SU(N_2)$ group (so that it only carries some overall $U(1)_B$ charge), while in the $U(N_2) \times U(N_2 + N_4)$ theory such an object naturally lives in the N_4 'th anti-symmetric product of fundamental representations of $SU(N_2 + N_4)$. Thus, we expect to need to have N_4 strings ending on this object.

Therefore, we expect

$$N_4 = \frac{1}{(2\pi l_s)^3 g_s} \int_{\mathbf{CP}^2} (-\hat{F}_4 - F \wedge F_2). \quad (3.39)$$

Recall that

$$\frac{1}{(2\pi l_s)^3 g_s} \int (-\hat{F}_4) = Q_4^{Page} = -k b_0. \quad (3.40)$$

¹⁷AH thanks Yuji Tachikawa for a conversation related to this issue.

Moreover, following the Freed-Witten argument presented in the previous section, we know that the brane wrapped on \mathbf{CP}^2 should have a *half*-integer gauge field flux on its worldvolume. The minimal possibility for the gauge flux is¹⁸ is

$$\frac{1}{(2\pi l_s)^3 g_s} \int (-F \wedge F_2) = \frac{k}{2}, \quad (3.41)$$

which suggests that b_0 must be

$$b_0 = -\frac{N_4}{k} + \frac{1}{2}. \quad (3.42)$$

Thus, the di-baryon vertex is consistent with our arguments above.

3.4.3 Baryon vertex

The D6-brane wrapped on \mathbf{CP}^3 was interpreted in [5] as a baryon vertex, on which N_2 strings corresponding to external sources in the fundamental representation of $U(N_2)$ can end. (Of course, we also expect to have an analogous object with $N_2 + N_4$ strings ending on it, but this is simply the bound state of the D6-brane with the D4-brane described above, or, equivalently, a D6-brane with one unit of gauge field flux on \mathbf{CP}^1 .) Since \mathbf{CP}^3 is a spin manifold, such a D6-brane does not experience the effects of the Freed-Witten anomaly, so the minimal energy configuration has a vanishing worldvolume flux. Denoting the worldvolume gauge potential by A , and its field strength by F , the terms in the worldvolume action responsible for determining the number of fundamental strings ending on the D6-brane are

$$\begin{aligned} S_{D6}^{WZ} &\sim - \int \left(A_7 + A_5 \wedge (F + B_2) + \frac{1}{2} A_3 \wedge (F + B_2) \wedge (F + B_2) \right. \\ &\quad \left. + \frac{1}{3!} A_1 \wedge (F + B_2) \wedge (F + B_2) \wedge (F + B_2) \right) \\ &= \int \left[-A_7 + A \wedge \left(F_6 + A_3 \wedge H_3 + B_2 \wedge F_4 + \frac{1}{2} B_2 \wedge B_2 \wedge F_2 \right) \right] \\ &= \int (-A_7 + A \wedge \hat{F}_6). \end{aligned} \quad (3.43)$$

where we dropped the terms proportional to H_3 , as they vanish. We recognize the D2-brane Page charge

$$Q_2^{Page} = N_2 + \frac{k}{12} \quad (3.44)$$

as giving rise to an anomalous worldvolume electric charge.

The contribution from the higher curvature terms, on the other hand, is given by

$$S_{D6}^R \sim - \int A_1 \wedge (F + B) \wedge \frac{1}{48} (p_1(N) - p_1(T))$$

¹⁸Other possible values of the flux correspond to bound states of these D4-branes with D2-branes wrapped on \mathbf{CP}^1 , which are vertices which have k strings ending on them [5].

$$= \int A \wedge F_2 \wedge \frac{1}{48}(-p_1(T) + p_1(N)). \quad (3.45)$$

We can evaluate this using

$$F_2 = 2\pi k g_s l_s J, \quad (3.46)$$

and

$$\int_{\mathbf{CP}^3} J \wedge p_1(T) = 4, \quad (3.47)$$

$$\int_{\mathbf{CP}^3} J \wedge p_1(N) = 0, \quad (3.48)$$

which are the periods over \mathbf{CP}^3 of the first Pontryagin classes of the tangent and normal bundles, respectively, wedged with the Kähler form J [46]. We obtain

$$\frac{1}{2\pi g_s l_s} \int_{\mathbf{CP}^3} F_2 \wedge \frac{1}{48}(-p_1(T) + p_1(N)) = -\frac{k}{12}. \quad (3.49)$$

This cancels the non-integer part of (3.44), giving rise to the net charge N_2 , which is precisely the expected number (and, in particular, it is an integer). This test is therefore unaffected by the shift in b_0 by $1/2$.

3.4.4 Parity

An important consistency check of our proposal is to show that the field theory and its gravity dual transform under parity in compatible ways.

In the absence of fractional branes, the ABJM superconformal field theory is invariant under the standard parity action of 3-dimensional gauge theory (reflecting one of the spatial coordinates, say x^2). Parity changes the sign of the two Chern-Simons terms, but because the gauge group is $U(N_2)_k \times U(N_2)_{-k}$ this maps the theory to itself. On the gravity side, M theory and type IIA string theory both admit a parity action, but this acts also as a change of sign for some of the background fields. In particular, in type IIA, parity takes $B_2 \rightarrow -B_2$. Taking $B_2 = (2\pi)^2 \alpha' b_0 J$ on the \mathbf{CP}^1 cycle in \mathbf{CP}^3 as above, an obvious parity-invariant choice is $b_0 = 0$. There is however an alternative choice, $b_0 = 1/2$. Under parity, this maps to $b_0 \rightarrow -1/2$, but since type IIA string theory is also invariant under $b_0 \sim b_0 + 1$, the choice $b_0 = 1/2$ is actually also invariant.

When fractional branes are included, the unequal ranks of the gauge groups imply that parity is not a symmetry of the theory, but rather it takes the theory into a different “parity-dual” theory, as pointed out in [6]. The inverse of the “slide transformation” (3.1) takes the $U(N_2)_k \times U(N_2 + N_4)_{-k}$ theory to a $U(N_2 + k - N_4)_k \times U(N_2)_{-k}$ theory, which was argued in [6] to be a dual description of the same SCFT, and the naive parity transformation then takes

this to a $U(N_2)_k \times U(N_2 + k - N_4)_{-k}$ theory. So, the parity transformation was conjectured to exchange $N_4 \rightarrow k - N_4$ without changing k .

In the gravity dual, it was argued in [6] that each fractional brane shifts b_0 by $1/k$, and it was assumed there that b_0 vanishes for $N_4 = 0$, leading to $b_0 = N_4/k$. Under parity (again acting as $x^2 \rightarrow -x^2$) this maps to $b_0 = -N_4/k$ which is equivalent by a large gauge transformation to $b_0 = (k - N_4)/k$, consistent with the field theory arguments above.

With the shifted quantization condition $b_0 = N_4/k - 1/2$, we see that the parity transformation $b_0 \rightarrow -b_0$ is again equivalent to sending $N_4 \rightarrow k - N_4$, this time without needing to perform any large gauge transformation shifting b_0 . This shows that our proposal still admits a natural (and possibly simpler) action of parity.

4 Duality cascades in $\mathcal{N} = 3$ Yang-Mills-Chern-Simons theories

The relation described above between the brane configurations of figure 9 suggests that perhaps different $U(N_2) \times U(N_2 + N_4)$ $\mathcal{N} = 3$ supersymmetric Yang-Mills-Chern-Simons theories could be equivalent at low energies. The simplest expectation may be that all of these theories flow to $U(N_2) \times U(N_2 + N_4)$ $\mathcal{N} = 6$ -superconformal Chern-Simons-matter theories, and that the superconformal theories related by (3.1) are all equivalent. However, it was argued in [6] that such superconformal theories exist only when $|N_4| \leq k$, since in other cases one can obtain on their moduli space (by moving out the N_2 “regular branes”) pure $\mathcal{N} = 3$ supersymmetric $U(N_4)_k$ theories with $|N_4| > k$, that are believed [35, 36, 48] to break supersymmetry. Thus, at most one cascade step of (3.1) can really relate $\mathcal{N} = 6$ superconformal theories, which is precisely the one described in section 3.4.4 above.

Nevertheless, we can still conjecture that when we have a $U(N_2 + N_4)_k \times U(N_2 + 2N_4 + k)_{-k}$ Yang-Mills-Chern-Simons theory with $0 < N_4 < k$, related by a single cascade step (3.1) to a $U(N_2)_k \times U(N_2 + N_4)_{-k}$ theory which is believed to flow to a $\mathcal{N} = 6$ superconformal fixed point, then this theory flows in the IR to the same fixed point. The original theory classically has an $8(N_2 + N_4)$ -dimensional moduli space, but, as mentioned above, assuming that this moduli space persists in the quantum theory leads to a contradiction. The conjecture above implies that the quantum moduli space is smaller, and is only $8N_2$ -dimensional. This has a nice interpretation using the modified s -rule discussed in the previous section. This s -rule tells us that the naive brane connections of the $U(N_2 + N_4) \times U(N_2 + 2N_4 + k)$ theory, which involve $(N_2 + N_4)$ regular branes going around the circle, do not lead to a supersymmetric configuration. However, there is another way to connect the branes, in which there are only N_2 regular branes (and the other N_4 regular branes join together with a fractional brane to give a fractional brane of higher winding number), and this connection does not violate the

s -rule (and, thus, is expected to lead to supersymmetric vacua). Similar arguments suggest that the $U(N_2+nN_4+n(n-1)k/2)_k \times U(N_2+(n+1)N_4+n(n+1)k/2)_{-k}$ theory may also flow to the same $\mathcal{N} = 6$ superconformal fixed point as the $U(N_2)_k \times U(N_2+N_4)_{-k}$ theory, for any integer n (when $|N_4| \leq k$), and that, correspondingly, it has a smaller quantum moduli space (which agrees with the modified s -rule). This flow involves a sequence of dualities reducing N_2 and N_4 , which are similar to Seiberg dualities, so the final picture would be very similar to the cascade of Klebanov and Strassler, except that here the cascade can smoothly end with a finite rank at high energies, and its IR end is given by a superconformal theory rather than a theory with a mass gap.¹⁹

Note that this conjecture still does not imply that all $U(N_2)_k \times U(N_2+N_4)_{-k}$ theories have supersymmetric vacua; when we start from $N_4 > k$ we can always try to perform cascade steps which are the inverse of (3.1), but when N_4 is too large we get to $N_2 \leq 0$ before we get to $N_4 < k$, and in these cases there is no possible superconformal field theory to flow to in the IR. Thus, we suggest that these cases (and, in particular, the $U(N_4)_k$ theory arising for $N_2 = 0$ when $N_4 > k$, as discussed in [35, 36, 48]) dynamically break supersymmetry²⁰. However, in some cases this breaking may happen in the IR after the theory first undergoes a series of duality cascades as described above.

Of course, the arguments for the duality cascade that we gave above were circumstantial and involved various assumptions and analogies. The cascade is consistent with all the usual consistency checks for supersymmetric dualities, once we assume that the moduli space of $\mathcal{N} = 3$ supersymmetric Yang-Mills-Chern-Simons theories can be modified in the quantum theory as described above. Since the field theories in question are always strongly coupled, it seems that the best way to test if such a cascade exists or not is to construct the dual string theory solutions, analogous to the one of [13], assuming that they are weakly curved and can be described by supergravity. This issue is currently under investigation [51].

¹⁹The picture of different ways to connect brane configurations on a circle, leading to different dimensional moduli spaces, may also be applied to the Klebanov-Strassler cascade [13, 49, 50]. The naive brane connection of the $SU(N) \times SU(N+M)$ theory in that case leads to a $6N$ -dimensional moduli space, but one could also connect M regular branes to M fractional branes to form M fractional branes winding once around the circle, or connect $2M$ regular branes to M fractional branes to form M fractional branes winding twice around the circle, etc. This suggests that the same theory may also have moduli spaces of dimensions $6(N-M)$, $6(N-2M)$, etc., and the existence of such branches of the moduli space is indeed confirmed by a careful analysis of the quantum moduli space of this theory [50]. Note that the match between these pictures is not precise, since in principle one could also connect just one regular brane to one fractional brane, and there is no corresponding branch of the moduli space found in [50]. Nevertheless, this example suggests that in some cases one can interpret different branches of the moduli space as corresponding to different brane connections, and we would like to claim that the same thing happens in our case.

²⁰For the theories with $N_2 > 0$ another possibility is that they have no vacuum, and have a runaway behavior along their classical moduli space.

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A Flux and potential quantization in the D1-D3 system

In this appendix we review the status of flux and charge quantization in the D1-D3 bound state system in type IIB string theory. Consider a system consisting of N_1 D1-branes extended along the 01 directions and N_3 D3-branes extended along the 0123 directions, with the 23 directions compactified on T^2 with some non-vanishing B_2 -field along this T^2 . We will take the D1-branes to have melted into the D3-branes, so that there is a non-vanishing F_{23} gauge field flux on the worldvolume of the D3-branes. With a non-zero B_{23} , one expects a shift in the D1-brane charge carried by the D3-branes, from F_{23} to $F_{23} + B_{23}$. The energy density of the branes is also shifted due to the presence of B_2 in the $\sqrt{-\det(G + B + F)}$ DBI action of the D3-branes, affecting their ADM mass.

To see the gravitational back-reaction of this, consider the type IIB supergravity solution for such a D1-D3 bound state [52]:

$$\begin{aligned} ds_{str}^2 &= f^{-1/2}[-dx_0^2 + dx_1^2 + h(dx_2^2 + dx_3^2)] + f^{1/2}(dr^2 + r^2 d\Omega_5^2) , \\ f &= 1 + \frac{\alpha'^2 R^4}{r^4} , \quad h^{-1} = \sin^2 \theta f^{-1} + \cos^2 \theta , \end{aligned} \tag{A.1}$$

$$\begin{aligned} B_{23} &= \left(B_\infty - \frac{\sin \theta}{\cos \theta} \right) + \frac{\sin \theta}{\cos \theta} f^{-1} h , \\ e^{2\phi} &= g_s^2 h , \quad F_{01r} = \frac{1}{g_s} \sin \theta \partial_r f^{-1} , \\ F_5 &= \tilde{F}_5 + * \tilde{F}_5 , \quad \tilde{F}_{0123r} = \frac{1}{g_s} \cos \theta h \partial_r f^{-1} . \end{aligned} \tag{A.2}$$

The parameters R and θ will be related to the charges below. We included in the solution a possible non-vanishing B_{23} at infinity by hand, which we expect to correspond to the

possibility of turning on a B_2 field on the torus asymptotically far away from the branes. At the level of solving the SUGRA equations of motion, this has no effect. On the other hand, one must explore how the charge quantization conditions are affected.

Computation of the self-dual 5-form flux in this background gives

$$\frac{1}{(2\pi l_s)^4} \int_{S^5} F_5 = \frac{1}{(2\pi l_s)^4 g_s} 4\alpha'^2 R^4 \cos \theta \Omega_5 = N_3, \quad (\text{A.3})$$

where Ω_5 is the volume of a unit 5-sphere, which implies

$$R^4 \cos \theta = 4\pi g_s N_3. \quad (\text{A.4})$$

With regards to the D1-brane charge, on the other hand, the modified equation of motion

$$d * F_3 = H \wedge F_5 \quad (\text{A.5})$$

implies that $*F_3$ is not closed and will not respect Gauss' law. However, the modification

$$\hat{F}_7 = *F_3 - B \wedge F_5 \quad (\text{A.6})$$

will give rise to a closed non-gauge-invariant 7-form. One can then either compute the Maxwell charge

$$Q_1^{\text{Maxwell}} = \frac{1}{(2\pi l_s)^5} \int_{T^2 \times S^5} *F_3 = \frac{N_3 R_1^2}{\alpha'} \sin \theta \quad (\text{A.7})$$

or the Page charge

$$Q_1^{\text{Page}} = \frac{1}{(2\pi l_s)^5} \int_{T^2 \times S^5} \hat{F}_7 = \frac{N_3 R_1^2}{\alpha'} (\sin \theta - B_\infty \cos \theta) = N_1, \quad (\text{A.8})$$

where R_1 refers to the radius of the x_2 and x_3 coordinates and is not to be confused with R , which is a parameter in f .

Combined with (A.4), this implies

$$R^4 = 4\pi g N_3 \sqrt{\frac{(B_\infty N_3 R_1^2 + \alpha' N_1)^2}{N_3^2 R_1^4} + 1}, \quad (\text{A.9})$$

$$\tan \theta = \frac{\alpha' N_1}{R_1^2 N_3} + B_\infty, \quad (\text{A.10})$$

which prescribes exactly how R and θ are discretized due to the discreteness of N_1 and N_3 which are integer-quantized. The Maxwell charge (A.7) with R and θ expressed in terms of N_1 and N_3 has the form

$$\frac{64\pi\alpha'^3}{R_1^2} \left(N_1 + \frac{B_\infty R_1^2}{\alpha'} N_3 \right). \quad (\text{A.11})$$

We see that the natural periodicity of B_∞ in units of α'/R_1^2 appear. A useful dimensionless parameter to define, therefore, is

$$\chi = \frac{R_1^2}{\alpha'} B_\infty. \quad (\text{A.12})$$

Suppose we now consider taking the near-horizon limit keeping χ ,

$$U \equiv \frac{r}{\alpha'}, \quad (\text{A.13})$$

and R_1 fixed. Then, the geometry approaches

$$\frac{ds^2}{\alpha'} = \frac{U^2}{R^2}(-dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{U^2}(dU^2 + U^2 d\Omega_5^2), \quad (\text{A.14})$$

with

$$R = 4\pi g_s N_3 \quad (\text{A.15})$$

and

$$B_{23} = -\frac{\alpha'}{R_1^2} \frac{N_1}{N_3}. \quad (\text{A.16})$$

Note that χ has decoupled from B_{23} , which now takes on values quantized in units of $\alpha'/R_1^2 N_3$. This shows that in the $AdS_5 \times S^5$ background (on a torus), B_{23} is not allowed to take arbitrary values at the quantum level. One can also think of the degree of freedom contained in χ as a singleton, which decouples in the limit where one focuses only on the $SU(N_3)$ part of the dynamics of the stack of D3-branes. Changes in B_{23} are locked with changes in N_1 through the presence of Chern-Simons terms, and the latter are only allowed to take on discrete values. Note also that these issues are strictly quantum and are invisible in the large N_3 limit.

Once we identify (A.6) as the Page charge, it is straightforward, via a chain of T-dualities, to obtain (2.32). The only difference between the D1-D3 system and the D6-brane wrapped on ALE is the orientation of the B_2 -field. Otherwise, the issues of induced charges and their quantization are identical.

One interesting question remains. How does one think of the Page charge (2.32) when lifted to M theory? The RR flux F_2 lifts in M-theory to a geometric flux. A natural candidate expression, borrowing the notation of [53, 54], is to write

$$\hat{G}_4 = G_4 + f \wedge C_3, \quad (\text{A.17})$$

where

$$f \wedge C = f_{bc}^a C_{ade} dx^b \wedge dx^c \wedge dx^d \wedge dx^e, \quad (\text{A.18})$$

and to use \hat{G}_4 to define the Page flux of the fractional M5-brane. It would be interesting to verify if this expression is correct.

B Conventions

B.1 Differential geometry conventions

Much of the material presented in this article depends sensitively on the details of sign conventions for Type II supergravity. In this appendix, we summarize the conventions which we follow.

- The signature of space-time is taken to be $(-, +, \dots, +)$.
- Differential forms are denoted $F_p = \frac{1}{p!} F_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$.
- Hodge duals are defined by

$$*F_p = \frac{1}{(D-p)!} F_{i_1 \dots i_p} \epsilon^{i_1 \dots i_p i_{p+1} \dots i_D} dx^{i_{p+1}} \wedge \dots \wedge dx^{i_D}, \quad (\text{B.1})$$

where

$$\sqrt{-\det g} \epsilon^{01 \dots (D-1)} = 1. \quad (\text{B.2})$$

- $|F|^2 \equiv F_{\mu_1 \dots \mu_p} F^{\mu_1 \dots \mu_p} = -F \wedge *F$.
- As a result of the definition of the Hodge dual operator $*$,

$$**F_p = -F_p \quad (\text{B.3})$$

for even p -form field strengths in type IIA supergravity.

B.2 Supergravity conventions

We consider the type IIA supergravity Lagrangian

$$S = \frac{1}{2\kappa_{10}^2} (S_{NSNS} + S_{RR} + S_{CS} + S_{DBI} + S_{WZ}), \quad (\text{B.4})$$

where the bulk terms are

$$S_{NSNS} = \int d^{10}x \sqrt{-g} e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} |H_3|^2 \right), \quad (\text{B.5})$$

$$S_{RR} = -\frac{1}{2} \int d^{10}x \sqrt{-g} \left(|F_2|^2 + |\tilde{F}_4|^2 \right), \quad (\text{B.6})$$

$$S_{CS} = \frac{1}{2} \int B_2 \wedge F_4 \wedge F_4, \quad (\text{B.7})$$

and the action on D-branes is

$$S_{DBI} = -(2\kappa_{10}^2 T_p) \int d^{p+1} \sqrt{-\det(g + B + F)}, \quad (\text{B.8})$$

$$S_{WZ} = -s \int A \wedge e^{B_2} \wedge *j_{p+1}^{Dp}, \quad (\text{B.9})$$

with $s = +1$ for a brane and $s = -1$ for an anti-brane. The constants and currents are normalized in terms of the string coupling and the string scale as

$$4\pi\kappa_{10}^2 = (2\pi l_s)^8 g_s^2 \quad (\text{B.10})$$

$$2\pi\kappa_{10}^2 T_p = (2\pi l_s)^{7-p} g_s \quad (\text{B.11})$$

$$j_{p+1}^{Dp} = (2\pi l_s)^{7-p} g_s \delta^{9-p}(\vec{y}) dx^0 \wedge dx^1 \dots \wedge dx^p, \quad (\text{B.12})$$

where \vec{y} denotes the transverse dimensions to the Dp-brane. This set of conventions has several implicit implications which we elaborate below.

- The type II supergravity Dp-branes with positive charge will have RR $p+1$ -form potential

$$A_{p+1} = -H^{-1} dx^0 \wedge \dots \wedge dx^{p+1}, \quad (\text{B.13})$$

where $H \sim r^{-(7-p)}$ and is positive for large r . An isolated brane will satisfy the linear relation

$$d * F_{p+1} = s * j_{p+1}^{Dp}. \quad (\text{B.14})$$

- We take the gauge-invariant type IIA 4-form to be

$$\tilde{F}_4 = dA_3 + dB_2 \wedge A_1. \quad (\text{B.15})$$

- The form of S_{WZ} implies that a D($p+2$)-brane in a presence of B_2 -field with positive $\int B_2$ will induce a positive Dp-brane charge.

- The presence of D4-branes and D6-branes implies that form fields A_5 and A_7 are implicitly part of the S_{WZ} term. They are understood as arising from generalizing the free part of S_{RR} so that

$$S_{RR} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g} (|F_2|^2 + |F_4|^2 + |F_6|^2 + |F_8|^2), \quad (\text{B.16})$$

but with the constraint

$$F_2 = *F_8, \quad F_4 = -*F_6. \quad (\text{B.17})$$

The constraint should not be viewed as being imposed directly on the action. Instead, note that the equations of motion

$$d * F_8 = s * j_7^{D6}, \quad (\text{B.18})$$

$$d * F_6 = s * j_5^{D4}, \quad (\text{B.19})$$

can be interpreted as imposing the following condition for the fluxes

$$\int_{S_2} F_2 = \int_{V_3} dF_2 = \int_{V_3} d * F_8 = s \int_{V_3} * j_7^{D6}, \quad (\text{B.20})$$

$$\int_{S_4} (-F_4) = \int_{V_5} d(-F_4) = \int_{V_5} d * F_6 = s \int_{V_5} * j_5^{D4}, \quad (\text{B.21})$$

with $S_i = \partial V_{i+1}$. The seemingly unnatural choice of signs in (B.17) can be motivated by considering the extension of the consideration above to type IIA supergravity, including the non-linear effects. They are constrained by the choice of the sign of the type IIA Chern-Simons term (B.7), the form of the gauge-invariant four-form (B.15), and the choice of the sign of B_2 in S_{WZ} (B.9), as can be seen from the following considerations.

The Maxwell equation and the Bianchi-identity for the 2-form and the 4-form field strengths take the form

$$dF_2 = *j_7^{brane} = s_6 *j_7^{D6}, \quad (B.22)$$

$$-d\tilde{F}_4 = *j_5^{brane} + H_3 \wedge F_2 = s_4 *j_5^{D4} + d(B_2 \wedge F_2), \quad (B.23)$$

$$d * \tilde{F}_4 = *j_3^{brane} - H_3 \wedge \tilde{F}_4 = s_2 *j_3^{D2} + d\left(B_2 \wedge (-\tilde{F}_4) - \frac{1}{2}B_2 \wedge B_2 \wedge F_2\right), \quad (B.24)$$

$$d * F_2 = *j_1^{brane} + H_3 \wedge * \tilde{F}_4 = s_0 *j_1^{D0} + d\left(B_2 \wedge (*\tilde{F}_4) - \frac{1}{2}B_2 \wedge B_2 \wedge (-\tilde{F}_4) + \frac{1}{6}B_2 \wedge B_2 \wedge B_2 \wedge F_2\right), \quad (B.25)$$

where $*j_{p+1}^{brane} = \sum_{q=p}^6 \left(s_q e^{B_2} \wedge *j_{q+1}^{Dq}\right)_{9-p}$ denotes the “brane charge” current, and s_q denotes the orientation of the D q -branes. If the surface integrals of the right-hand sides of the above expressions are to be interpretable as the supergravity manifestation of induced charges due to S_{WZ} , the only consistent identification is (B.17).

In this convention, the D-brane charges are computed as a surface integral

$$Q_6^{Page} = \int F_2, \quad (B.26)$$

$$Q_4^{Page} = \int (-\tilde{F}_4) - B_2 \wedge F_2, \quad (B.27)$$

$$Q_2^{Page} = \int * \tilde{F}_4 - B_2 \wedge (-\tilde{F}_4) + \frac{1}{2}B_2 \wedge B_2 \wedge F_2, \quad (B.28)$$

$$Q_0^{Page} = \int *F_2 - B_2 \wedge (*\tilde{F}_4) + \frac{1}{2}B_2 \wedge B_2 \wedge (-\tilde{F}_4) - \frac{1}{6}B_2 \wedge B_2 \wedge B_2 \wedge F_2. \quad (B.29)$$

The relations (B.23) and (B.25) are closely tied to the form of the gauge-invariant four-form (B.15). The relation (B.24), on the other hand, is connected to the sign of the Chern-Simons term (B.7). Had the sign of the Chern-Simons term (B.7) been switched, it would have been natural, instead, to use opposite signs in (B.17).

- The gauge-invariant 4-form (B.15) arises from dimensional reduction of the M-theory 4-form by the ansatz

$$C_3 = A_3 - B_2 \wedge dx^{11}, \quad (B.30)$$

and lifts to the 11-dimensional action

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{2}|G_4|^2 \right) - \frac{1}{2\kappa_{11}^2} \int \frac{1}{6}C_3 \wedge G_4 \wedge G_4. \quad (B.31)$$

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